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FLAT CONNECTIONS, GEOMETRIC INVARIANTS AND THE SYMPLECTIC NATURE OF THE FUNDAMENTAL GROUP OF SURFACES

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In this paper we associate a new geometric invariant to the space of flat connections on a G $(= \mathrm{SU}(2))$-bundle on a compact Riemann surface M and relate it to the symplectic structure on the space $\mathrm{Hom}(\pi_1(M), G)/G$ consisting of representations of the fundamental group $\pi_1(M)$ of M into G modulo the conjugate action of G on representations.

Introduction. Our setup is as follows. Let $G = \mathrm{SU}(2)$ and M be a compact Riemann surface and $E \to M$ be the trivial G-bundle. (Any $\mathrm{SU}(2)$-bundle over M is topologically trivial.) Let $\mathscr{C}$ (resp. $\mathscr{C}^*$) be the space of all (resp. irreducible) connections and $\mathscr{F}$ (resp. $\mathscr{F}^*$) the subspace of all (resp. irreducible) flat connections on this G-bundle. We put the Fréchet topology on $\mathscr{C}$ and the subspace topology on $\mathscr{F}$.

Given a loop $\sigma: S^1 \to \mathscr{F}$, we can extend σ to the closed unit disc $\tilde{\sigma}: D^2 \to \mathscr{C}$, since $\mathscr{C}$ is contractible. On the trivial G-bundle $E \times D^2 \to M \times D^2$ we define a "tautological" connection form ϑ_σ as follows.

$$\vartheta_\sigma|_{(e,t)} = \tilde{\sigma}(t) \quad \forall\, (e, t) \in E \times D^2.$$

Clearly restriction of ϑ_σ to the bundle $E \times \{t\} \to M \times \{t\}$ is $\tilde{\sigma}(t)$ $\forall\, t \in D^2$. Let $K(\theta_\sigma)$ be the curvature form of ϑ_σ. Evaluation of the second Chern polynomial on this curvature form $K(\vartheta_\sigma)$ gives a closed 4-form on $M \times D^2$, which when integrated along D^2 yields a 2-form on M. This 2-form is closed since $\dim M = 2$ and thus defines an element in $H^2(M, \mathbb{R}) \approx \mathbb{R}$. It is seen that this class is independent of the extension of σ. We thus have a map

$$\chi: \Omega(\mathscr{F}) \to H^2(M, \mathbb{R}) \approx \mathbb{R}$$

where $\Omega(\mathscr{F})$ is the loop space of $\mathscr{F}$.

It is seen that χ induces a map

$$\overline{\chi}: \Omega(\mathscr{F}^*/\mathscr{G}) \to \mathbb{R}/\mathbb{Z}$$

where $\mathscr{G} = \mathrm{Map}(M, G)$ is the gauge group of the G-bundle $E \to M$.