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ERRATA TO: THE SET OF PRIMES DIVIDING THE LUCAS NUMBERS HAS DENSITY 2/3

J. C. LAGARIAS

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Theorem C of my paper [2] states an incorrect density for the set of primes that divide the terms W_n of a recurrence of Laxton [3], due to a slip in the proof. A corrected statement and proof are given.

The corrected version of Theorem C of [2] is:

THEOREM C. *Let W_n denote the recurrence defined by $W_0 = 1$, $W_1 = 2$ and $W_n = 5W_{n-1} - 7W_{n-2}$. Then the set*

$$S_W = \{p\colon p \text{ is prime and } p \text{ divides } W_n \text{ for some } n \geq 0\}$$

has density $3/4$.

The proof below proceeds along the general lines of §4 of [2].

Proof. One has

$$W_n = \left(\frac{3+\sqrt{-3}}{6}\right)\left(\frac{5+\sqrt{-3}}{2}\right)^n + \left(\frac{3-\sqrt{-3}}{6}\right)\left(\frac{5-\sqrt{-3}}{2}\right)^n.$$

If

$$\alpha = \frac{3+\sqrt{-3}}{6} \quad \text{and} \quad \phi = \frac{5+\sqrt{-3}}{5-\sqrt{-3}} = \frac{11+5\sqrt{-3}}{14}$$

then

$$W_n \equiv 0 \pmod{p} \Leftrightarrow \phi^n \equiv -\frac{\overline{\alpha}}{\alpha} \pmod{(p)} \quad \text{in } \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right],$$

where $-\frac{\overline{\alpha}}{\alpha} = \frac{-1+\sqrt{-3}}{2}$ is a cube root of unity. Consequently

(1.1) $\qquad p$ divides W_n for some $n \geq 0 \Leftrightarrow \operatorname{ord}_{(p)}\phi \equiv 0 \pmod{3}$.

The argument now depends on whether the prime ideal (p) splits or remains inert in the ring of integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ of $\mathbb{Q}(\sqrt{-3})$.

Case 1. $p \equiv 1 \pmod{3}$, so that $p = \pi\overline{\pi}$ in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. Since $\operatorname{ord}_{(\pi)}\phi = \operatorname{ord}_{(\overline{\pi})}\phi$, one has

$$\operatorname{ord}_{(p)}\phi \equiv 0 \pmod{3} \Leftrightarrow \operatorname{ord}_{(\pi)}\phi \equiv 0 \pmod{3}.$$