PACIFIC JOURNAL OF MATHEMATICS
Vol. 162, No. 2, 1994

IDEALS OF FINITE CODIMENSION IN FREE ALGEBRAS AND THE FC-LOCALIZATION

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The topology defined by all finite codimensional right ideals has interesting properties in the case of the free algebra and the group ring of a free group. Its open ideals are precisely the finitely generated essential ones. Finitely generated right ideals are closed and the Leavitt numbers of the associated localization are 1 and $n-1$. The proofs are, for the most part, applications of Schreier's method.

This research started when we first became acquainted with the (known) fact that the maximal ring of quotients of a free ring does not have the unique rank property, i.e. there are free modules over it that are of different finite ranks yet they are isomorphic. In fact, since the free ring, and the group ring of a free group have essential right ideals that are free of (countably) infinite rank, it turned out that over the maximal ring of quotients any two finitely generated free modules are isomorphic (see §5 below). But we noticed also that all the examples we had of finitely generated essential ideals, either in the free ring on n generators over a field K or in KG with G a free group of rank n, are of rank $1+m(n-1)$. Our examples were all derived from subgroups of finite index (where Schreier's formula holds!) or by the fractal method, as in our paper **[9]**. Again, it turns out to be known, and due to Lewin **[8]**, that Schreier's formula extends to ideals of finite codimension in the free ring or the free group ring over a field. From here it is natural to conjecture that if one imitates the construction of the maximal ring of quotients, $Q_{\max}$, but uses right ideals of finite codimension only, then the result would be a ring with the property that two free modules over it are isomorphic if and only if their ranks are congruent modulo $n-1$. This localization we call Q_{fc}. It is defined as $L(L(R))$ where

$$L(R)=\varinjlim\{\operatorname{Hom}_R(I, R) \mid I \text{ is finite-codimensional in } R\}.$$

Note that a right ideal of finite codimension in the group ring KG is an analogue (and generalization) of the notion of subgroups of finite index in G: to a subgroup H of finite index in G corresponds the right ideal of KG generated by the augmentation ideal of KH.