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A CONVEXITY THEOREM FOR SEMISIMPLE SYMMETRIC SPACES

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In this paper we prove a convexity theorem for semisimple symmetric spaces which generalizes Kostant's convexity theorem for Riemannian symmetric spaces. Let τ be an involution on the semisimple connected Lie group G and $H = G_0^\tau$ the 1-component of the group of fixed points. We choose a Cartan involution θ of G which commutes with τ and write $K = G^\theta$ for the group of fixed points. Then there exists an abelian subgroup A of G, a subgroup M of K commuting with A, and a nilpotent subgroup N such that $HMAN$ is an open subset of G and there exists an analytic mapping $L: HMAN \to \mathfrak{a} = \mathbf{L}(A)$ with $L(hman) = \log a$. The set of all elements in A for which $aH \subseteq HMAN$ is a closed convex cone. Our main result is the description of the projections $L(aH) \subseteq \mathfrak{a}$ for these elements as the sum of the convex hull of the Weyl group orbit of $\log a$ and a certain convex cone in $\mathfrak{a}$.

0. Introduction. If G is a connected semisimple Lie group and $G = KA'N$ an Iwasawa decomposition, then the convexity theorem of Kostant describes the image of the sets aK under the projection $G = KA'N \to \mathfrak{a}' = \mathbf{L}(A')$, $k \exp Xn \mapsto X$ as the convex hull of the Weyl group orbit through $\log a$. Recently van den Ban proved a generalization of this theorem to the following situation. Let τ be an involution on the semisimple Lie group G with finite center, $G = KA'N$ a compatible Iwasawa decomposition, i.e., K is τ-invariant, and $\mathfrak{a}' = \mathfrak{a}_{\mathfrak{h}} + \mathfrak{a}_{\mathfrak{q}}$ the corresponding decomposition of $\mathfrak{a}' = \mathbf{L}(A')$ into 1 and -1 eigenspaces for τ. Suppose that $H \subseteq G^\tau$ is an essentially connected subgroup (see §I for the definition). Then he describes the image of the sets aH, $a \in \exp \mathfrak{a}_{\mathfrak{q}}$ under the projection $F: G \to \mathfrak{a}_{\mathfrak{q}}$ defined by $g \in K \exp(\mathfrak{a}_{\mathfrak{h}}) \exp F(g) N$. This set is the sum of the convex hull of the orbit of $\log a$ under a certain Weyl group and a convex cone in $\mathfrak{a}_{\mathfrak{q}}$.

We generalize Kostant's theorem into another direction. We consider the projection $L: HMAN \to \mathfrak{a}$ defined by $g \in HM \exp L(g) N$, where $H \subseteq G^\tau$ is essentially connected and M, A, and N are defined in §I. This makes sense because the A-component in a product $hman$ is unique and $HMAN$ is open in G. So the main new difficulties are the non-compactness of H and the fact that the projection L