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ON FACTOR REPRESENTATIONS OF DISCRETE RATIONAL NILPOTENT GROUPS AND THE PLANCHEREL FORMULA

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The purpose of this paper is to extend the Kirillov orbit picture of representation theory for nilpotent Lie groups to discrete groups $G_{\mathbf{Q}}$ defined over the rationals $\mathbf{Q}$, following a program begun by Roger Howe. Let Ad^* be the coadjoint action of $G_{\mathbf{Q}}$ on the Pontryagin dual $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ of the Lie algebra of $G_{\mathbf{Q}}$. It is shown that each coadjoint orbit closure is a coset of the annihilator of an ideal of $\mathfrak{g}_{\mathbf{Q}}$, that a certain induced representation canonically associated with an orbit closure is a traceable factor, and that there is an orbital integral formula which gives the trace. Finally, a Plancherel formula is proved.

1. Introduction. In what follows, let $\mathfrak{g}_{\mathbf{Q}}$ be a nilpotent Lie algebra with rational structure constants; i.e., $\mathfrak{g}_{\mathbf{Q}}$ has a basis $\{X_1, \dots, X_n\}$ of vectors such that

$$[X_i, X_j] = \sum_{k=1}^{n} \alpha_{i,j,k} X_k,$$

with all $\alpha_{i,j,k} \in \mathbf{Q}$. Then we form a nilpotent group $G_{\mathbf{Q}}$ with base set $\mathfrak{g}_{\mathbf{Q}}$, using the Campbell-Baker-Hausdorff formula to define a polynomial group multiplication, and we have a map $\exp: \mathfrak{g}_{\mathbf{Q}} \to G_{\mathbf{Q}}$, $\exp \equiv \mathrm{Id}$ on the set $\mathfrak{g}_{\mathbf{Q}}$. We will think of $\mathfrak{g}_{\mathbf{Q}}$ and $G_{\mathbf{Q}}$ as having the discrete topology. $G_{\mathbf{Q}}$ has no normal abelian group of finite index; therefore $G_{\mathbf{Q}}$ is not of Type I ([**Tho**]). The purpose of this paper is to develop an extension of the Kirillov orbit picture of representation theory for nilpotent Lie groups to discrete groups $G_{\mathbf{Q}}$ defined over the rationals, following a program begun by Roger Howe ([**How1**], [**How2**], [**How3**]). In particular, in [**How2**], he constructs an extension of the Kirillov theory to finitely generated discrete nilpotent groups without torsion; this work was a chief source of inspiration to us.

Although it is in general difficult to work with non-type-I groups (and in particular discrete groups lack some of the structure that non-type-I connected Lie groups G possess: see [**Puk**], [**Gdt**]), it is possible to say a great deal in the present case. Let $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ denote the Pontryagin dual of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}$; we examine the structure of closures of