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PRODUCTIVE POLYNOMIALS

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The problem addressed is: When is a class B of polynomials in n non-commuting indeterminates closed under substitution into a given polynomial q ?

1. Introduction. Let $\mathbb{F}$ be a field and let $\mathbb{F}\langle x_1, \dots, x_n\rangle$ be the linear algebra of polynomials in the non-commuting indeterminates $x_1, \dots, x_n$. Let $q \in \mathbb{F}\langle x_1, \dots, x_n\rangle$. Let A be an associative[1] algebra over $\mathbb{F}$. q defines a mapping $\hat{q}$ of $A \times \cdots \times A = A^n$ into A whose value $\hat{q}(a_1, \dots, a_n)$ at $(a_1, \dots, a_n)$ is the result of replacing each x_i in q by the corresponding a_i, and then carrying out the algebraic operations proper to A. A linear subspace B of the algebra A will be called q-closed if whenever $\mathbf{A} = (a_1, \dots, a_n) \in A^n$ then $\hat{q}(\mathbf{a}) \in B$. Let $q((B))$ be the smallest q-closed linear subspace containing B. We study mainly the case that A is $\mathbb{F}\langle x_1, \dots, x_n\rangle$ itself, and B is the linear subspace generated by $x_1, \dots, x_n$ and the unit 1. The q-closed set generated by $x_1, \dots, x_n$ and 1 will be denoted in this case simply by $((q))$.

We will usually use just P to stand for $\mathbb{F}\langle x_1, \dots, x_n\rangle$. $q \in P$ will be called *productive* if $((q)) = P$; and otherwise, *non-productive.*

Two questions interest us:

1.1. When is a given $q \in P$ productive,
and

1.2. If it is not, how to find elements p which are not in $((q))$?

A clear-cut answer to 1.1 is given by 3.9. An answer to 1.2 is given in §4, illustrated by an example 8.5. We regard q as an n-ary operation and prepare a suitable ideal theory.

2. Theorems establishing productivity. Consider $q = x_1x_2$. Then a linear subspace B is q-closed if it contains the product of any pair of members: B is a subalgebra.[2] Thus, if B is the linear subspace generated by $x_1, \dots, x_n$ and 1, then $((q))$ is the algebra generated by $x_1, \dots, x_n$ and 1. This being P, x_1x_2 is productive.

[1] In this paper, all algebras are supposed to be associative, and so we omit the term.

[2] Note that B is x_1x_2-closed if and only if it is x_ix_j-closed, where i, j are any two distinct indices.