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EXTREMAL FUNCTIONS AND THE CHANG-MARSHALL INEQUALITY

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Answering a question of J. Moser, S.-Y. A. Chang and D. E. Marshall proved the existence of a constant C such that $\frac{1}{2\pi}\int_0^{2\pi} e^{|f(e^{i\theta})|^2}\,d\theta \le C$ for all functions f analytic in the unit disk with $f(0)=0$ and Dirichlet integral not exceeding one. We show that there are extremal functions for the functionals $\Lambda_\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta$ when $0 \le \alpha < 1$. We establish a variational condition satisfied by extremal functions. We show that the identity function $f(z) = z$ is a local maximum in a certain sense for the functionals Λ_α and conjecture that it is a global maximum.

1. Introduction. The Dirichlet space $\mathfrak{D}$ consists of those functions f analytic on the unit disk Δ which have finite Dirichlet integral

$$\|f\|_{\mathfrak{D}}^2 = \frac{1}{\pi}\iint_\Delta |f'(z)|^2\,dx\,dy.$$

We will always assume that $f(0)=0$. It is well-known and easy to establish that $\mathfrak{D}$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\iint_\Delta f'(z)\overline{g'(z)}\,dx\,dy,$$

and that, if $f(z) = \sum_{n=1}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty b_n z^n$, then

$$\langle f, g\rangle = \sum_{n=1}^\infty n a_n \overline{b_n}.$$

In particular,

$$\|f\|_{\mathfrak{D}}^2 = \sum_{n=1}^\infty n|a_n|^2.$$

For $\alpha \ge 0$ and $f \in \mathfrak{D}$, we define

$$\Lambda_\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta.$$

This is known to be finite for all $\alpha \ge 0$ and all $f \in \mathfrak{D}$, and it can be shown that the quantity

$$I_\alpha = \sup\{\,\Lambda_\alpha(f) \mid \|f\|_{\mathfrak{D}} \le 1\,\}$$