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ON COMPLETE RIEMANNIAN MANIFOLDS WITH COLLAPSED ENDS

ZHONGMIN SHEN

We show that if a complete open manifold with bounded curvature and sufficiently small ends, then each end is an infranilend. Conversely, an open manifold with finitely many infranilends admits a complete metric with bounded curvature and arbitrarily small ends.

1. Introduction. It is well known that if a complete open manifold M of finite volume has bounded negative sectional curvature, i.e.

$$-\Lambda_2 \leq \sec(M) \leq -\Lambda_1,$$

where Λ_i are positive constants, then M has finite topological type (see **[G1]**). In particular, M has finitely many ends. Moreover, each end collapses, i.e., for any end E and any point $p \in M$,

$$\lim_{r\to\infty} \operatorname{diam}(E \cap S(p, r)) = 0,$$

where $S(p, r) = \{x \in M; d(p, x) = r\}$ denotes the geodesic sphere of radius r around p. Further, each end is topologically of the form $N\times(0, \infty)$ for some infranilmanifold N. See **[E]** and **[Sc]** for details. See also **[K]** in the case $\Lambda_2/\Lambda_1 < 4$.

An open manifold M is said to have N ends, if there is a compact subset K such that for any compact subset $K \subset K' \subset M$, $M\backslash K'$ contains exactly N unbounded components. Simply we call any such component an end of M.

In **[Sh]** we studied complete open Riemannian manifolds M with sectional curvature bounded from below and small ends. In order to state the result, we need to introduce some notations. For $r > 0$, the connected components, Σ, of $\partial(M\backslash\overline{B(p, r)})$, are called the boundary components, where $B(p, r)$ denotes the open geodesic ball of radius r around p. Following **[C]** (compare **[AG]**), we define the essential diameter $\mathscr{D}(p, r)$ at distance r from p by

$$(p, r) = \sup_{\Sigma} \operatorname{diam}(\Sigma),$$

where the supremum is taken over all boundary components Σ of $M\backslash\overline{B(p, r)}$ with $\Sigma \cap R(p, r) \neq \varnothing$, where $R(p, r) = \{\gamma(r); \gamma$ *is a ray from* $p\} \subset S(p, r)$. Notice that in the definition of $\mathscr{D}(p, r)$ we do not