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TWO-POINT DISTORTION THEOREMS FOR UNIVALENT FUNCTIONS

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We establish a one-parameter family of symmetric, linearly invariant two-point distortion theorems for univalent functions defined on the unit disk. The weakest theorem in the family is a symmetric, linearly invariant form of a classical distortion theorem of Koebe, while another special case is a distortion theorem of Blatter. All of these distortion theorems are necessary and sufficient for univalence. Each of these distortion theorems can be expressed as a two-point comparison theorem between euclidean and hyperbolic geometry on a simply connected region; however, none of these comparison theorems characterize simply connected regions. We obtain analogous results for convex univalent functions and convex regions, except that in this context the two-point comparison theorems do characterize convex regions.

1. Introduction. We begin by recalling some basic information about the hyperbolic metric and related material. The hyperbolic metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is given by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

It is normalized to have constant Gaussian curvature -4. A region Ω in the complex plane $\mathbb{C}$ is called hyperbolic if $\mathbb{C}\backslash\Omega$ contains at least two points. The density of the hyperbolic metric on a hyperbolic region Ω is obtained from

$$\lambda_{\Omega}(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

where $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection of $\mathbb{D}$ onto Ω. The density is independent of the choice of the covering projection of $\mathbb{D}$ onto Ω. The hyperbolic metric on Ω induces the hyperbolic distance function d_{Ω} as follows:

$$d_{\Omega}(a, b) = \inf \int_{\gamma} \lambda_{\Omega}(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining a and b. The infimum is actually a minimum; there always exists a path δ in