PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

MÖBIUS-INVARIANT HILBERT SPACES IN POLYDISCS

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We define the Dirichlet space $\mathscr{D}$ on the unit polydisc $\mathbb{U}^n$ of $\mathbb{C}^n$. $\mathscr{D}$ is a semi-Hilbert space of of holomorphic functions, contains the holomorphic polynomials densely, is invariant under compositions with the biholomorphic automorphisms of $\mathbb{U}^n$, and its semi-norm is preserved under such compositions. We show that $\mathscr{D}$ is unique with these properties. We also prove $\mathscr{D}$ is unique if we assume that the semi-norm of a function in $\mathscr{D}$ composed with an automorphism is only equivalent in the metric sense to the semi-norm of the original function. Members of a subclass of $\mathscr{D}$ given by a norm can be written as potentials of $\mathscr{L}^2$-functions on the n-torus $\mathbb{T}^n$. We prove that the functions in this subclass satisfy strong-type inequalities and have tangential limits almost everywhere on $\partial\mathbb{U}^n$. We also make capacitory estimates on the size of the exceptional sets on $\partial\mathbb{U}^n$.

1. Introduction. *Möbius-invariant spaces.* Let $\mathbb{U}$ be the open *unit disc* in $\mathbb{C}$ and $\mathbb{T}$ be the *unit circle* bounding it. The open *unit polydisc* $\mathbb{U}^n$ and the *torus* $\mathbb{T}^n$ in $\mathbb{C}^n$ are the cartesian products of n unit discs and n unit circles, respectively. $\mathbb{T}^n$ is the distinguished boundary of $\mathbb{U}^n$ and forms only a small part of the topological boundary $\partial\mathbb{U}^n$ of $\mathbb{U}^n$. We denote by $\mathscr{M}$ the group of all *biholomorphic automorphisms* of $\mathbb{U}^n$ (the *Möbius group*). The subgroup of *linear* automorphisms in $\mathscr{M}$ is denoted by $\mathscr{U}$. The space of holomorphic functions with domain $\mathbb{U}^n$ will be called $\mathscr{H}(\mathbb{U}^n)$ and will carry the topology of *uniform convergence on compact subsets* of $\mathbb{U}^n$.

A *semi-inner product* on a complex vector space $\mathscr{H}$ is a sesquilinear functional on $\mathscr{H} \times \mathscr{H}$ with all the properties of an inner product except that it is possible to have $\langle\langle a, a\rangle\rangle = 0$ when $a \neq 0$. $\|a\| = \sqrt{\langle\langle a, a\rangle\rangle}$ is the associated *semi-norm*. We assume $\langle\langle \cdot, \cdot\rangle\rangle$ is not identically zero.

DEFINITION 1.1. $\mathscr{H}$ is called a *Hilbert space* of holomorphic functions on $\mathbb{U}^n$ if

(i) $\mathscr{H}$ is a linear subspace of $\mathscr{H}(\mathbb{U}^n)$,
(ii) the semi-inner product $\langle\langle \cdot, \cdot\rangle\rangle$ of $\mathscr{H}$ is complete,
(iii) $\mathscr{H}$ contains all (holomorphic) polynomials,
(iv) polynomials are dense in $\mathscr{H}$ in the topology of the semi-norm $\|\cdot\|$ of $\mathscr{H}$.