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ON THE FROBENIUS MORPHISM OF FLAG SCHEMES

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Dedicated to Professor C. W. Curtis on the occasion of his 65th birthday

We give a new proof to V. B. Mehta and A. Ramanthan's theorem that the Schubert subschemes in a flag scheme are all simultaneously compatibly split, using the representation theory of infinitesimal algebraic groups. In particular, the present proof dispenses with the Bott-Samelson schemes.

Let K be a perfect field of positive characteristic p. If A is a K-algebra and $r \in \mathbf{Z}$, one defines a new K-algebra $A^{(r)}$ by the ring homomorphism $K \to A$ such that $\xi \mapsto \xi^{p^{-r}}$. Given a K-scheme $\mathfrak{X}$ we will denote by $\mathfrak{X}^{(r)}$ the K-scheme having the same underlying topological space as that of $\mathfrak{X}$ but with the structure sheaf $\mathscr{O}_{\mathfrak{X}} \otimes_K K^{(-r)}$, which we regard as a sheaf of K-algebras by the usual multiplication of K on $K^{(-r)}$ from the right. If $\mathscr{F}$ is an $\mathscr{O}_{\mathfrak{X}}$-module, we set $\mathscr{F}^{(r)} = \mathscr{F} \otimes_K K^{(-r)}$; it comes equipped with the structure of an $\mathscr{O}_{\mathfrak{X}^{(r)}}$-module. If $r > 0$, the morphism $F_{\mathfrak{X}}^r \colon X \to X^{(r)}$ that is the identity on the underlying topological spaces and such that $a \otimes \xi \mapsto a^{p^r}\xi$ for each $a \in \Gamma(\mathfrak{V}, \mathscr{O}_{\mathfrak{X}})$ and $\xi \in K^{(-r)}$ with $\mathfrak{V}$ open in $\mathfrak{X}$ is called the rth Frobenius morphism of $\mathfrak{X}$.

If K is algebraically closed, Hartshorne **[HASV]**, (III.6.4) showed that on the projective spaces over K, the direct image of any invertible sheaf under the Frobenius morphism splits into a direct sum of invertible sheaves; this was crucial for B. Haastert **[Haas]** to prove the $\mathscr{D}$-affinity of the projective spaces. We will compute in §1 which invertible sheaf enters as a direct summand.

More generally, we say after V. B. Mehta and A. Ramanathan **[MR]** that $\mathfrak{X}$ is Frobenius split iff the structural morphism $F_{\mathfrak{X}}^{\mathfrak{f}} \colon \mathscr{O}_{\mathfrak{X}^{(1)}} \to F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$ admits a left inverse, called a Frobenius splitting, so that $\mathscr{O}_{\mathfrak{X}^{(1)}}$ is a direct summand of $F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$. If σ is a Frobenius splitting of $\mathfrak{X}$ and if $\mathfrak{Y}$ is a closed subscheme of $\mathfrak{X}$ defined by an ideal sheaf $\mathscr{I}$, we say σ splits $\mathfrak{Y}$ iff $\sigma(F_{\mathfrak{X}*}\mathscr{I}) \subseteq \mathscr{I}^{(1)}$, in which case $\mathfrak{Y}$ will also be Frobenius split, said to be compatibly split in $\mathfrak{X}$.

Mehta and Ramanathan showed that the flag schemes are Frobenius split with all the Schubert subschemes compatibly split. Their