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LATTICES OF LIPSCHITZ FUNCTIONS

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Let M be a metric space. We observe that $\mathrm{Lip}(M)$ has a striking lattice structure: its closed unit ball is lattice-complete and completely distributive. This motivates further study into the lattice structure of $\mathrm{Lip}(M)$ and its relation to M. We find that there is a nice duality between M and $\mathrm{Lip}(M)$ (as a lattice). We also give an abstract classification of all normed vector lattices which are isomorphic to $\mathrm{Lip}(M)$ for some M.

The set $\mathrm{Lip}(M)$ of bounded real-valued Lipschitz functions on a metric space M has been studied extensively (see [2] for some references) as either a Banach space or a Banach algebra. However, its natural lattice structure has been almost completely ignored, probably because it is not a Banach lattice: the "Riesz norm" law

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|,$$

which connects lattice structure with norm, is not satisfied by either of the two norms customarily given to $\mathrm{Lip}(M)$. (Here $|x| = x \vee (-x)$.)

Nonetheless, the lattice structure of $\mathrm{Lip}(M)$ is intimately related to its most natural norm. Indeed, for any norm-bounded set of elements $\{x_\alpha\} \subset \mathrm{Lip}(M)$, the join $\bigvee x_\alpha$ exists and satisfies

$$(*) \qquad \left\|\bigvee x_\alpha\right\| \leq \sup\{\|x_\alpha\|\}.$$

Since $-\bigvee x_\alpha = \bigwedge(-x_\alpha)$ (whenever either side exists), this implies a similar statement for meets and is equivalent to saying that the closed unit ball of $\mathrm{Lip}(M)$ is lattice-complete. What's more, the unit ball is completely distributive, which makes it very special from the lattice-theoretic point of view. We therefore feel that a study of $\mathrm{Lip}(M)$ which emphasizes its lattice structure is well warranted.

This paper begins such a study. Having identified the special lattice properties of $\mathrm{Lip}(M)$, we also find it interesting to examine the class of all normed vector lattices which share these properties. We will call these objects *Lip-spaces*. (Note: by "normed vector lattice" we simply mean a vector lattice which is equipped with a vector space norm. This is at variance with a usage of this term which requires that the norm be a Riesz norm.)