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THE INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS FOR LOCALLY FREE ACTIONS

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Let a connected unimodular Lie group G act smoothly and locally freely on a closed manifold X. Assume that the isotropy groups of the action are torsion-free. Let K be the maximal compact subgroup of G. Let T be a G-invariant first order differential operator on X that is elliptic in directions transverse to the G-orbits. Using Kasparov products over C^*G, we prove index formulas equating indices of elliptic operators on $K\backslash X$ with linear combinations of multiplicities of G-representations in kernel(T) − kernel(T^*).

Introduction. Let a connected unimodular Lie group G act smoothly on a closed manifold X. Let T be a G-invariant first order differential operator on X that is elliptic in directions transverse to the G-orbits. Kernel(T) and kernel(T^*) need not be finite-dimensional, but they are direct sums of irreducible G-representations, each occurring with finite multiplicity. (We work with assumptions, described in §2, that guarantee that we have Hilbert space structures and unitary G-representations as needed.) The following is then an interesting index problem. For each irreducible G-representation π, calculate the difference:

$$\text{multiplicity of } \pi \text{ in } \ker(T) - \text{multiplicity of } \pi \text{ in } \ker(T^*) .$$

M. Atiyah and I. Singer studied the index theory of invariant operators elliptic in directions transverse to the orbits of a compact Lie group action [**At1**]. They phrased the index problem as the computation of a distribution on G. Let α^+, respectively α^-, be the representation of G on $\ker(T)$, respectively $\ker(T^*)$. The index distribution is then the functional on $C^\infty(G)$ defined for $f \in C^\infty(G)$ by

$$f \to \operatorname{Tr}\left(\int_G f(g)\alpha^+(g)\,dg\right) - \operatorname{Tr}\left(\int_G f(G)\alpha^-(G)\,dg\right) .$$

M. Vergne has now given a formula for this distribution in a neighborhood of the identity [**Ve**]. The foundations of this approach to the index problem extend to noncompact G [**Sin**] [**NeZi**].

In this paper we focus on the direct calculation of the difference of multiplicities when G acts locally freely. For a locally free action