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A COUNTER-EXAMPLE CONCERNING THE PRESSURE IN THE NAVIER-STOKES EQUATIONS, AS $t \to 0^+$

JOHN G. HEYWOOD AND OWEN D. WALSH

We show the existence of solutions of the Navier-Stokes equations for which the Dirichlet norm, $\|\nabla \mathbf{u}(t)\|_{L^2(\Omega)}$, of the velocity is continuous as $t = 0$, while the normalized L^2-norm, $\|p(t)\|_{L^2(\Omega)/R}$, of the pressure is not. This runs counter to the naive expectation that the relative orders of the spatial derivatives of $\mathbf{u}$, p and $\mathbf{u}_t$ should be the same in a priori estimates for the solutions as in the equations themselves.

1. Introduction. We consider the initial boundary value problem for the Navier-Stokes equations in an open bounded domain $\Omega \subset R^n$ $(n = 2$ or $3)$, with $\partial\Omega \in C^2$:

$$(1)\quad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \Delta \mathbf{u} - \nabla p, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{for } x \in \Omega \text{ and } t > 0,$$
$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 .$$

For reference, let

$$D(\Omega) = \{\varphi \in C_0^\infty(\Omega) \colon \nabla \cdot \varphi = 0\},$$
$$J(\Omega) = \text{ completion of } D(\Omega) \text{ in the } L^2(\Omega)\text{-norm } \|\cdot\|,$$
$$J_1(\Omega) = \text{ completion of } D(\Omega) \text{ in the Dirichlet-norm } \|\nabla \cdot\|.$$

It is well known that if $\mathbf{u}_0 \in J_1(\Omega)$, then $\mathbf{u} \in C([0, T); J_1(\Omega))$ and

$$(2)\qquad \|\nabla \mathbf{u}(t)\|^2 + \int_0^t [\|\mathbf{u}(s)\|^2_{W_2^2(\Omega)} + \|\nabla p(s)\|^2 + \|\mathbf{u}_t(s)\|^2]\, ds \le C\|\nabla \mathbf{u}_0\|^2, \qquad 0 < t < T,$$

where T and C can be expressed as constants depending only on $\|\nabla \mathbf{u}_0\|$ and Ω (we are not concerned here with their optimal values).

It seems natural to expect that the relative orders of spatial differentiation of $\mathbf{u}$, p and $\mathbf{u}_t$ should be the same in a priori estimates for the Navier-Stokes equations as in the equations themselves. That is, p should appear with one less spatial derivative than $\mathbf{u}$, and $\mathbf{u}_t$ with two less, as they do under the integral sign in (2) and in many