PACIFIC JOURNAL OF MATHEMATICS
Vol. 164, No. 2, 1994

# CONJUGATE POINTS ON SPACELIKE GEODESICS OR PSEUDO-SELF-ADJOINT MORSE-STURM-LIOUVILLE SYSTEMS

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**This paper develops the basic theory of conjugate points along geodesics in manifolds with indefinite metric; equivalently, that of conjugate points for Morse-Sturm-Liouville systems which are symmetric with respect to an indefinite inner product. The theory is rather different from that for Riemannian manifolds or that for timelike or null geodesics in Lorentzian manifolds. We find that conjugate points may be unstable with respect to perturbation of the geodesic: they may annihilate in pairs. Also the conjugate points need not be isolated: we construct an example where a whole ray is conjugate to a given point. Nevertheless, we give an extension of the Morse Index Theorem to this situation. We also analyze the effects of certain perturbations.**

**1. Introduction.** The study of the length functional on Riemannian manifolds is fundamental to both classical and modern differential geometry. Classically, of course, the stationary points of this functional are the geodesics. The modern exploitation began with the Morse Index Theorem, which identified the index of the second variation (the "number of decreasing directions") with the algebraic count of the number of conjugate points along the geodesic. Morse himself used developments of this theory to prove deep results about the existence of periodic geodesics on the two-sphere equipped with an arbitrary metric [**11**]; Bott was led by a similar analysis of Lie groups to his celebrated Periodicity Theorem [**4**].

For Lorentzian manifolds, the existence of conjugate points on null or timelike geodesics has physical significance. For timelike geodesics, there is an effect rather like the "twin paradox," except that no accelerations are involved; for null geodesics, one has the phenomenon of gravitational lensing. Conjugate points along both these geodesics play a role in the singularity theorems of Penrose and Hawking, as well. (See [**13**] for a review.) The Morse Index Theorem and its consequences for these cases were established by Beem and Ehrlich [1–3]. We refer to the timelike and null cases collectively as *causal.*

Although the *interpretation* of the Morse Index Theorem is different