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# THE BOUNDARY DISTORTION OF A QUASICONFORMAL MAPPING

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**We investigate how quasiconformal mappings $f: \mathbf{B}^n \to \mathbf{R}^n$ distort Hausdorff measures of sets on the boundary $\partial \mathbf{B}^n$.**

**1. Introduction.** Recent exciting developments in the study of the boundary behavior of conformal mappings, orchestrated by Makarov (**[M1]**, **[M2]**, **[M3]**), lead to the natural question: what is the boundary distortion of a quasiconformal mapping $f$ of the unit ball $\mathbf{B}^n$ into $\mathbf{R}^n$? Makarov's results assert that for any conformal mapping of the unit disk in the plane there is a set of full measure on the boundary whose image has Hausdorff dimension precisely one. There is no hope for such results for general quasiconformal mappings as shown by well-known examples. In fact, given $0 < \alpha < 1$ there is a quasiconformal self-homeomorphism of the unit disk carrying a set of full measure on the circle into a set of Hausdorff dimension $\alpha$ (see **[Ro]**, **[T2]**). In the other direction, given $1 < \alpha < 2$ it is not difficult to construct a quasiconformal mapping of the disk onto a Jordan domain such that the image of any boundary set of positive measure has Hausdorff dimension $\alpha$.

In this article we discuss various questions related to the boundary distortion of quasiconformal mappings; in particular, we demonstrate that in spite of the discouraging counterexamples at least some features of the restricted expansion/contraction phenomenon are retained. Naturally, our results lag behind the deep information available in the case of conformal mappings, but we feel that some of the techniques used in this paper are of interest in the higher dimensional quasiconformal theory. It is also our hope that the modest beginning here will inspire future research in this area. Several open questions are listed at the end of the paper.

**2. Main results.** In this section we describe our main results. The proofs will follow in subsequent sections.

2.1. *Notation.* We let $B(x, r)$ stand for the open $n$-ball centered at $x$ with radius $r$, and we assume $n \geq 2$. For short, $B_r = B(0, r)$