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PARTIALLY MEASURABLE SETS IN MEASUR SPACES

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1. Introduction. This article is concerned with any sets, whether measurable or not, in a general measure space (a general measure space is briefly defined in the article). A set S has an interior measure $m_i(S)$ and an exterior measure $m_e(S)$, where $0 \leq m_i(S) \leq m_e(S)$. Consider two disjoint sets S_1 and S_2, i.e. $S_1 \cap S_2 = \emptyset$, and their union set $S_1 \cup S_2$. There are the six non-negative quantities $m_i(S)$ and $m_e(S)$, for $S = S_1, S_2$, and $S_1 \cup S_2$. It is well-known that $m_e(S)$ is subadditive, i.e. $m_e(S_1 \cup S_2) \leq m_e(S_1) + m_e(S_2)$, and $m_i(S)$ is superadditive, i.e. $m_i(S_1 \cup S_2) \geq m_i(S_1) + m_i(S_2)$. The present article obtains more inequalities on the six quantities $m_i(S)$ and $m_e(S)$ for $S = S_1, S_2$, and $S_1 \cup S_2$; and indeed obtains a specific collection of six linear inequalities. One of these six inequalities states that the average measure $\frac{1}{2}(m_i(S) + m_e(S))$ is subadditive. Also, average measure is countably subadditive. Further, if the general measure space satisfies a certain two conditions, it is shown that these six inequalities form a *complete* collection of *independent* inequalities, valid for every pair of disjoint sets S_1, S_2. These two condition on the measure space do hold for the usual measure spaces.

Previous articles of the author considered the same matters for Lebesgue measure on the real number line or in Euclidean n-dimensional space. These are listed as [**1**] and [**2**] in the References at the end of the present article. The present article extends these results to a general measure space, subject to some limitations.

The inequalities on $m_e(S)$ and $m_i(S)$, for $S = S_1, S_2$, and $(S_1 \cup S_2)$, where $S_1 \cap S_2 = \emptyset$, are stated in Theorems 2 and 4 below, in §3. Furthermore, if the measure space satisfies two conditions, the partitionable condition and the basis condition, these inequalities are shown to be a *complete* collection of inequalities which are valid for every pair of disjoint sets contained in a measurable set M.