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THE DISTRIBUTION MOD n OF FRACTIONS WITH BOUNDED PARTIAL QUOTIENTS

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Given a reduced fraction c/d with $0 < c < d$, there is a unique continued fraction expansion of c/d as $[0; a_1, a_2, \dots a_r]$ with $r \geq 1$, $0 \leq a_j$ for $1 \leq j \leq r$, and $a_r \geq 2$. For fixed positive integer n, the asymptotic distribution of the pair (c, d) mod n among the $n^2 \prod_{p|n}(1 - 1/p^2)$ possible pairs of congruence classes is uniform when averaged over the set $Q(x) := \{(c, d) : 0 < c < d \leq x, \gcd(c, d) = 1\}$ as $x \to \infty$. The main result is that if attention is restricted to the (rather thin) subset $Q_m(x)$ of relatively prime pairs (c, d) so that all the continued fraction convergents $a_j \leq m$, the same equidistribution holds. As a corollary, the relative frequency, both in $Q(x)$ and in $Q_m(x)$ for any fixed $m > 1$, of reduced fractions (c, d) so that $d \equiv b$ mod n, is asymptotic to $n^{-1} \prod_{p|gcd(b,n)}(1 - p^{-1}) \prod_{q|n}(1 - q^{-2})^{-1}$. These results lend further heuristic support to Zaremba's conjecture, which in this terminology reads that for some m (perhaps even $m = 2$) the set of denominators d occurring in $Q_m(x)$ includes all but finitely many natural numbers. The proofs proceed from some recent estimates for the asymptotic size of $Q_m(x)$. Thereafter, the argument is combinatorial.

1. Introduction. Among fractions c/d with $0 \leq c < d$, gcd $(c, d) = 1$, and $d \leq x$, asymptotically equal proportions have $(c, d) \equiv (0, 1)$ mod 2, $(c, d) \equiv (1, 0)$ mod 2, and $(c, d) \equiv (1, 1)$ mod 2. (The proof is immediate and is left to the reader.) The same equidistribution among classes $(a \bmod n, b \bmod n)$ with gcd (gcd (a, n), gcd $(b, n)) = 1$ holds by a fairly simple inclusion and exclusion calculation. Numerical experimentation and Occam's razor both suggest that the same equidistribution should hold when attention is restricted to fractions c/d of the form $[0; a_1, a_2, \dots a_r]$ with $a_r > 1, r \geq 1$ and all $a_i \leq m$. So it is. But before giving