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GENERALIZATION OF THE HILBERT METRIC TO THE SPACE OF POSITIVE DEFINITE MATRICES

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We introduce a generalization of the Hilbert projective metric to the space of positive definite matrices which we view as part of the Lagrangian Grassmannian.

1. Introduction. In his treatment of Kalman Bucy filters Bougerol [**1**], [**2**] uses the Riemannian metric on the set of positive definite matrices considered as a Riemannian symmetric space.

Graphs of symmetric linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ are Lagrangian subspaces in the standard linear symplectic space $\mathbb{R}^n \times \mathbb{R}^n$. We call a Lagrangian subspace positive, if it is a graph of a positive definite linear map. Further, we call a linear symplectic map monotone, if it maps positive Lagrangian subspaces onto positive Lagrangian subspaces. Bougerol discovered that the symplectic matrices in Kalman filtering theory are monotone. He shows that the action of any monotone map on the manifold of positive Lagrangian subspaces contracts the metric of the Riemannian symmetric space. It is the only (up to scale) Riemannian metric which has this property.

The goal of this paper is to introduce a natural Finsler metric in the manifold of positive definite matrices which, in addition to being contracted by the action of any monotone map, has striking geometric properties. In particular, we obtain that the coefficient of least contraction is equal to the hyperbolic tangent of one half of the diameter of the image. This is the same relation which was obtained by Birkhoff [**3**](see also [**4**]) for the Hilbert projective metric.

In the case of the positive orthant the Hilbert metric is also only Finsler (cf. [**6**]), which reflects the nonsmoothness of the cone. It is natural that the generalization of the Hilbert metric to the space of positive definite matrices is not smooth, because its boundary in the Lagrangian Grassmannian is not smooth.