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EXISTENCE OF SHORTEST DIRECTED NETWORKS IN $\mathbb{R}^2$

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This paper establishes the existence of a shortest directed network connecting a given set of points. In such networks, up to six segments sometimes meet at a point.

1. Introduction.

The standard Steiner problem considers shortest undirected networks, and at most three segments meet at a point ([**CR**, pp. 354-361], [**BG**], [**M1**], [**M2**]).

1.1. Definitions. A *directed network* is a finite system of one-way roads (oriented straight line segments) connecting all of a given set of starting points to all of a given set of ending points. (See Figure 1.1.) We refer to the given starting and ending points as *boundary points.* The *nodes* are any other points where the segments meet. We require that boundary points and nodes occur only at the endpoints of segments. Two segments meeting at a point count as one node. For $m \geq 3$, m segments meeting at a point count as $m-2$ nodes. When counting the number of edges meeting at a point, a *double edge* (an edge with both orientations) counts once, although its length counts twice. A *region* is the closure of a bounded component of the complement of the network.

1.2. Existence of length-minimizing directed networks in $\mathbb{R}^2$. The difficulty in demonstrating the existence of shortest directed networks lies primarily in obtaining an upper bound on the number of nodes in the network. In previous studies of networks in other settings, the number of nodes in the networks could be easily estimated since their minimizing networks obviously contained no cycles [**Ab**], [**A4**], [**L**]. Unfortunately, shortest directed networks may contain cycles. The number of cycles containing at least one boundary point may be estimated by the number of boundary points. Since shortest directed networks may have cycles containing no boundary