ON TOTALLY DIFFERENTIABLE AND SMOOTH FUNCTIONS

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1. Introduction. H. Rademacher has proved that a function of n variables satisfying a Lipschitz condition is totally differentiable a.e. (almost everywhere) (see, for instance, Saks, [6, pp. 310-311]). It was discovered by H. Federer (though not stated as a theorem; see [2, p. 442]) that if f is totally differentiable a.e. in the bounded set P, then there is a closed set $Q \subset P$ with the measure $|P - Q|$ as small as desired, such that f is smooth (continuously differentiable) in Q; that is, the values of f in Q may be extended through space so that the resulting function g is smooth there.

Theorem 1 of the present paper strengthens the latter theorem by showing that f is approximately totally differentiable a.e. in P if and only if Q exists with the above property. The rest of the paper gives further theorems in the direction of Federer's Theorem, as follows.

Suppose the domain of definition of f were a bounded open set P. Then in applying the part (a) $\longrightarrow$ (c) of Theorem 1, we might alter f in a set $P - Q$ which included a neighborhood of the boundary of P. In applications, it might be important to keep the values of f in most of a subset close to the boundary of P, or in most of some other subset. That such can be done follows from Theorem 2.

If f satisfies a Lipschitz condition, Theorem 3 shows that g may be made to satisfy a Lipschitz condition also, with a constant which equals a number ρ_n (depending on the number n of variables only) times the constant for f; in the case of one variable, we may take $\rho_1 = 1$.

If we weaken the assumption on f, assuming only that it is measurable, then Lusin's Theorem shows that we can alter f on a set of arbitrarily small measure, giving a continuous function g. In the other direction, suppose we assume that f (defined in an open set) has continuous mth partial derivatives, and that these derivatives are totally differentiable a.e. Then Theorem 4 shows that we may alter f on a set of arbitrarily small measure, giving a function g which has continuous partial derivatives of order $m + 1$. For the case of one variable, this is essentially a theorem of Marcinkiewicz, [5, Theorem 3].

Examples show that the hypotheses in the theorems cannot be materially

Received February 20, 1950.

Pacific J. Math. 1 (1951), 143-159.