THE ASYMPTOTIC EXPANSION OF A RATIO OF GAMMA FUNCTIONS

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1. Introduction. Many problems in mathematical analysis require a knowledge of the asymptotic behavior of the quotient $\Gamma(z+\alpha)/\Gamma(z+\beta)$ for large values of $|z|$. Examples of such problems are the study of integrals of the Mellin-Barnes type, and the investigation of the asymptotic behavior of confluent hypergeometric functions when the variable and one of the parameters become very large simultaneously.

Stirling's series can be used to find a first approximation for our quotient for very large $|z|$, it being understood that α and β are bounded. Without too much algebra one finds

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O(|z|^{-2})\right] \tag{1}$$

as $z \to \infty$, under conditions which will be stated later; but the determination of the coefficients of z^{-2}, z^{-3}, $\cdots$, in the asymptotic expansion of which (1) gives the first two terms, is a very laborious process, and the determination of the general term from Stirling's series is a well-nigh hopeless task.

The present paper originated when the first-named author (F.G. Tricomi) noticed that the asymptotic expansion of $\Gamma(z+\alpha)/\Gamma(z+\beta)$ can be obtained by methods similar to those which he used in a recent investigation of the asymptotic behavior of Laguerre polynomials [3]. The first proof given in this paper, and the detailed investigation of the coefficients A_n and C_n, are entirely due to him. Afterwards, the second named author (A. Erdélyi) pointed out that a shorter proof can be given by using Watson's lemma. His contributions to the present paper are the second proof, the generating function (18) of the coefficients, and their expression in terms of generalized Bernoulli polynomials.

We may mention that the same quotient was recently investigated by J.S. Frame [1]; but there is no overlapping with the results presented here.

Received April 17, 1950 and, with the contribution of A. Erdélyi, July 24, 1950. This research was sponsored by the Office of Naval Research.

Pacific J. Math. 1 (1951), 133-142.