# ON THE DEFINITION OF NORMAL NUMBERS

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**1. Introduction.** Let $R$ be a real number with fractional part $.x_1x_2x_3\cdots$ when written to scale $r$. Let $N(b,n)$ denote the number of occurrences of the digit $b$ in the first $n$ places. The number $R$ is said to be *simply normal* to scale $r$ if

$$\lim_{n\to\infty}\frac{N(b,n)}{n}=\frac{1}{r} \tag{1}$$

for each of the $r$ possible values of $b$; $R$ is said to be *normal* to scale $r$ if all the numbers $R, rR, r^2R, \cdots$ are simply normal to all the scales $r, r^2, r^3, \cdots$. These definitions, for $r = 10$, were introduced by Émile Borel [1], who stated (p. 261) that "la propriété caractéristique" of a normal number is the following: that for any sequence $B$ whatsoever of $v$ specified digits, we have

$$\lim_{n\to\infty}\frac{N(B,n)}{n}=\frac{1}{r^v}, \tag{2}$$

where $N(B,n)$ stands for the number of occurrences of the sequence $B$ in the first $n$ decimal places.

Several writers, for example Champernowne [2], Koksma [3, p. 116], and Copeland and Erdös [4], have taken this property (2) as the definition of a normal number. Hardy and Wright [5, p. 124] state that property (2) is equivalent to the definition, but give no proof. It is easy to show that a normal number has property (2), but the implication in the other direction does not appear to be so obvious. If the number $R$ has property (2) then any sequence of digits

$$B = b_1b_2\ \cdots b_v$$

appears with the appropriate frequency, but will the frequencies all be the same for $i = 1, 2, \cdots, v$ if we count only those occurrences of $B$ such that $b_1$ is an $i, i+v, i+2v, \cdots$ -*th* digit? It is the purpose of this note to show that this is

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