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INDEFINITE KAC-MOODY ALGEBRAS OF SPECIAL LINEAR TYPE

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From the special linear Lie algebra $A_n = s\ell(n+1,\mathbb{C})$ we construct certain indefinite Kac-Moody Lie algebras $IA_n(a,b)$ and then use the representation theory of A_n to determine explicit closed form root multiplicity formulas for the roots α of $IA_n(a,b)$ whose degree satisfies $|deg(\alpha)| \leq 2a+1$. These expressions involve the well-known Littlewood-Richardson coefficients and Kostka numbers. Using the Euler-Poincaré Principle and Kostant's formula, we derive two expressions, one of which is recursive and the other closed form, for the multiplicity of an arbitrary root α of $IA_n(a,b)$ as a polynomial in Kostka numbers.

Introduction.

For Kac-Moody algebras the root multiplicities of only the finite and affine algebras are explicitly known. In this paper, the third of a series of articles on the structure of non-finite, non-affine Kac-Moody algebras, we study certain indefinite Kac-Moody algebras coming from the special linear Lie algebra $A_n = s\ell(n+1,\mathbb{C})$ of traceless $(n+1)\times(n+1)$ complex matrices. The main theme of these articles is that combinatorial results from the representation theory of classical simple Lie algebras can be applied to the problem of determining root multiplicities for Kac-Moody algebras. The starting point is a well-known construction of graded Lie algebras of Kac-Moody type whose ingredients are a Lie algebra G over $\mathbb{C}$, two G-modules V and V', and a G-module homomorphism $\psi : V' \otimes V \longrightarrow G$. The graded Lie algebra $\mathcal{L} = \mathcal{L}(G,V,V',\psi) = \sum_{k\in\mathbb{Z}} \mathcal{L}_k$ built from these components contains no graded ideals which intersect the local part $V \oplus G \oplus V'$ trivially. The algebra G is specialized to be $g\ell(n+1,\mathbb{C}) = s\ell(n+1,\mathbb{C}) \oplus \mathbb{C}I$. The G-module V is assumed to be $V(b\Lambda_1) = V(b\epsilon_1)$, the irreducible G-module with highest weight b times the first fundamental weight Λ_1, or equivalently b times ϵ_1, where ϵ_1 maps a matrix to its $(1,1)$ entry. The homomorphism ψ is the map given by (2.1) below. A certain parameter "a" enters into the definition of ψ. We argue that the algebra $\mathcal{L}(G,V,V^*,\psi)$ is isomorphic to the Kac-Moody algebra having generalized Cartan matrix