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PERMUTATION MODEL FOR SEMI-CIRCULAR SYSTEMS AND QUANTUM RANDOM WALKS

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We give an approximation model for semi-circular families of non-commutative random variables, which is based on the permutation group. This model is compared to the random matrix model of Voiculescu, and a common refinement of the two approximations by explicit matrices with $0-1$ entries is exhibited.

Introduction.

The recent theory of semi-circular systems, introduced by D.Voiculescu [**16**] has proved to be a very efficient tool in the theory of free group factors (see e.g. [**13**] and the references therein), mainly due to the fact that semi-circular systems can be approximated by gaussian random matrices [**17**]. It is thus an interesting problem to try to find other simple approximations of semi-circular systems. In this article we introduce such an approximation, which is based on permutation groups.

Let us describe briefly the key fact which is at the basis of this approximation. Let $\mathfrak{S}_{n+1}$ be the permutation group of $\{0,1,\dots,n\}$, and for $1 \leq k \leq n$, let $\tau_k \in \mathfrak{S}_{n+1}$ be the transposition which exchanges 0 and k, and denote again by τ_k the corresponding left translation operator on $L^2(\mathfrak{S}_{n+1})$, then the trace of the spectral measure of the operator $\frac{1}{\sqrt{n}}(\tau_1+\dots+\tau_n)$ converges, as n goes to infinity, towards the semi-circle distribution. This fact is a consequence of Theorem 1 below.

In the second part of the paper, we show how this permutation model is related to the gaussian random matrix model of Voiculescu. We first describe a convenient way of obtaining a gaussian hermitian matrix, which is to let a brownian motion evolve on the set of hermitian matrices. This brownian motion was first studied by Dyson, after Wigner's work on the semi-circle law. We show that the evolution of the spectrum of a brownian hermitian matrix is very much linked to the theory of tensor product representations of the unitary groups. This enables us to approximate in law a hermitian random matrix by explicit finite dimensionnal matrices. This gives us a family of finite dimensionnal matrices depending on two integer parameters n and d. The random matrix model of Voiculescu is obtained when we let