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VALUES OF BERNOULLI POLYNOMIALS

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Dedicated to Emma Lehmer

Let $B_n(t)$ be the nth Bernoulli polynomial. We show that $B_{p-1}(a/q) - B_{p-1} \equiv q(U_p - 1)/2p \pmod p$, where U_n is a certain linear recurrence of order $[q/2]$ which depends only on a, q and the least positive residue of $p \pmod q$. This can be re-written as a sum of linear recurrence sequences of order $\leq \phi(q)/2$, and so we can recover the classical results where $\phi(q) \leq 2$ (for instance, $B_{p-1}(1/6) - B_{p-1} \equiv (3^p - 3)/2p + (2^p - 2)/p \pmod p$). Our results provide the first advance on the question of evaluating these polynomials when $\phi(q) > 2$, a problem posed by Emma Lehmer in 1938.

Introduction.

It has long been known that the nth Bernoulli polynomial $B_n(t)$, where

$$B_n(t) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} t^j$$

and B_k, the kth Bernoulli number, defined by the power series

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!} ,$$

take 'special' values at certain rational numbers with small denominators:

$$B_n(1) = B_n(0) = B_n, \quad \text{for } n \neq 1 \tag{1}$$
$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n;$$

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