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ON QUADRATIC RECIPROCITY OVER FUNCTION FIELDS

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A proof of quadratic reciprocity over function fields is given using the inversion formula of the theta function.

Over the years, many authors have produced proofs of the law of quadratic reciprocity. In 1857, Dedekind [**2**] stated that quadratic reciprocity holds over function fields; this was later proved by Artin [**1**]. One of the simplest proofs over the rational numbers relies on the functional equation of the classical theta function (see, for example, [**3**]); this technique was later generalized by Hecke [**4**] to number fields. In this note we use an analogous technique to give a simple and direct proof of quadratic reciprocity over rational function fields. We thank David Grant for suggesting this application of Theorem 2.3 of [**6**].

The reader is referred to [**5**] for a more complete discussion of the history of the Law of Quadratic Reciprocity.

Let $\mathbf{F} = \mathbf{F}_p$ be a finite field with p elements; for the sake of clarity we assume p is an odd prime. Let T be an indeterminate, and set $\mathbf{A} = \mathbf{F}[T]$. Then for $\alpha, \beta \in \mathbf{A}$ with α irreducible, let

$$\left(\frac{\beta}{\alpha}\right) = \begin{cases} 1 & \text{if } \beta \text{ is a (nonzero) quadratic residue modulo } \alpha, \\ -1 & \text{if } \beta \text{ is a (nonzero) quadratic nonresidue modulo } \alpha, \\ 0 & \text{if } \alpha \text{ divides } \beta. \end{cases}$$

We will show that for $\alpha, \beta \in \mathbf{A}$ distinct monic irreducible polynomials,

$$\left(\frac{\beta}{\alpha}\right) = \begin{cases} \left(\frac{-1}{p}\right)\left(\frac{\alpha}{\beta}\right) & \text{if } \deg\alpha, \deg\beta \text{ are both odd,} \\ \left(\frac{\alpha}{\beta}\right) & \text{otherwise.} \end{cases}$$

We require the following definitions.

Let $\mathbf{K} = \mathbf{F}(T)$; let $\mathbf{K}_\infty$ denote the completion of $\mathbf{K}$ with respect to the "infinite" valuation $|\cdot|_\infty$ given by $|\alpha/\beta|_\infty = p^{\deg\alpha - \deg\beta}$ where $\alpha, \beta \in \mathbf{A}$. (We adopt the convention that $\deg 0 = -\infty$, and hence $|0|_\infty = 0$.) One easily sees that $\mathbf{K}_\infty = \mathbf{F}\left(\left(\frac{1}{T}\right)\right)$, formal Laurent series in $\frac{1}{T}$; for $x \in \mathbf{K}_\infty$, we write $x = \sum_{j=-\infty}^{n} x_j T^j$. The "unit ball" or "ring of integers" in $\mathbf{K}_\infty$ is