PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

COHERENT STATES, HOLOMORPHIC EXTENSIONS, AND HIGHEST WEIGHT REPRESENTATIONS

KARL-HERMANN NEEB

Let G be a connected finite dimensional Lie group. In this paper we consider the problem of extending irreducible unitary representations of G to holomorphic representations of certain semigroups S containing G and a dense open submanifold on which the semigroup multiplication is holomorphic. We show that a necessary and sufficient condition for extendability is that the unitary representation of G is a highest weight representation. This result provides a direct bridge from representation theory to coadjoint orbits in $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. Namely the moment map associated naturally to a unitary representation maps the orbit of the highest weight ray (the coherent state orbit) to a coadjoint orbit in $\mathfrak{g}^*$ which has many interesting geometric properties such as certain convexity properties and an invariant complex structure.

In this paper we use the interplay between the orbit picture and representation theory to obtain a classification of all irreducible holomorphic representations of the semigroups S mentioned above and a classication of unitary highest weight representations of a rather general class of Lie groups. We also characterize the class of groups and semigroups having sufficiently many highest weight representations to separate the points.

0. Introduction.

A closed convex cone W in the Lie algebra $\mathfrak{g}$ is called *invariant* if it is invariant under the adjoint action. The starting point in the theory of holomorphic extensions of unitary representations was Ol'shanskiĭ's observation that if W is a pointed generating invariant cone in a simple Lie algebra $\mathfrak{g}$, where G is a corresponding linear connected group, and $G_{\mathbb{C}}$ its universal complexification, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbb{C}}$ ([**Ol82**]). This theorem has been generalized by Hilgert and 'Olafsson to solvable groups ([**HiOl92**]) and the most general result of this type, due to Lawson ([**La94**], [**HiNe93**]), is that if $G_{\mathbb{C}}$ is a complex Lie group with an antiholomorphic involution inducing the complex conjugation on $\mathfrak{g}_{\mathbb{C}} = \mathbf{L}(G_{\mathbb{C}})$, then the set