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SMALL EIGENVALUE VARIATION AND REAL RANK ZERO

OLA BRATTELI AND GEORGE A. ELLIOTT

A necessary and sufficient condition, in terms of asymptotic properties of the sequence, is given for the inductive limit of a sequence of finite direct sums of matrix algebras over commutative C*-algebras to have real rank zero (i.e., for each self-adjoint element to be approximible by one with finite spectrum).

1. Introduction.

In this paper we will consider C*-algebra inductive limits $A = \varinjlim A_n$ where the C*-algebras A_n have the form

$$A_n = \bigoplus_{j=1}^{r_n} \mathrm{C}(\Omega_{n,j}\, , \mathrm{M}_j)\ , \tag{1.1}$$

with each $\Omega_{n,j}$ a compact metrizable space (possibly empty, but let us assume that at least Ω_{n,r_n} is non-empty), r_n finite and M_j the C*-algebra of $j \times j$ complex matrices.

(We could just as well consider non-compact spaces, but the resulting increased generality in Theorem 1.1, below, would be illusory — one could reduce easily to the compact case.)

The spectrum Ω_n of A_n is the disjoint union of the spaces $\Omega_{n,j}$, $j = 1, \cdots, r_n$. If Λ is a clopen (closed and open) subset of Ω_n, let $A_n(\Lambda)$ denote the corresponding sub-C*-algebra of A_n, i.e.,

$$A_n(\Lambda) = \bigoplus_{j=1}^{r_n} \mathrm{C}(\Omega_{n,j} \cap \Lambda\, , \mathrm{M}_j)\ .$$

If Λ_1 is a clopen subset of Ω_m, let $\Phi_{(m,\Lambda_1)(n,\Lambda)} = \Phi_{\Lambda_1\Lambda}$ denote the homomorphism $A_n(\Lambda) \to A_m(\Lambda_1)$ obtained from $\Phi_{m,n}$ by cutting down by the unit of $A_m(\Lambda_1)$ in A_m and restricting to $A_n(\Lambda)$. Let $\mathcal{P}(\Omega_n)$ denote the set of partitions of Ω_n into clopen sets which are refinements of the partition

$$\{\Omega_{n,1}, \Omega_{n,2}, \cdots, \Omega_{n,r_n}\}\ .$$