PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 1, 1996

HOMOGENEOUS RICCI POSITIVE 5-MANIFOLDS

D. ALEKSEEVSKY, ISABEL DOTTI AND C. FERRARIS

We classify all 5-dimensional homogeneous Riemannian manifolds with positive Ricci curvature and among these we determine all Einstein manifolds. A new Einstein metric is found.

1. Introduction.

It is well known that a 2 or 3 dimensional Einstein manifold is of constant curvature. On the other hand it was proved by Jensen [**J1**] that a 4-dimensional homogeneous Einstein manifold is symmetric. In dimension 5, Wang and Ziller showed that $S^2 \times S^3$ admits infinitely many $S^3 \times S^3$-invariant and non-isometric Einstein metrics with positive scalar curvature ([**W-Z**]). The purpose of this paper is to classify the 5-dimensional homogeneous Riemannian manifolds with positive Ricci curvature and to determine those which are Einstein (see [**B**], 7.4.2 and following statements). We will first show that this problem reduces to studying the $S^3 \times S^3$-invariant metrics on $S^3 \times S^3/S^1$, by analyzing the various possibilities (see §3).

The $S^3 \times S^3$ invariant metrics in $S^3 \times S^3/S^1$ split naturally into two types, diagonal and non diagonal. For every imbedding of S^1 into $S^3 \times S^3$ there exists a unique Einstein metric of diagonal type. They coincide (see Remark 2) with those considered in [**W-Z**]. When the isotropy representation is multiplicity free only the diagonal type occurs (see Proposition 4.1). This family was also considered by E.Rodionov in [**R**]. When the isotropy representation has equivalent subrepresentations both types of metrics (diagonal and non-diagonal) occur. We find that, in the class of non-diagonal metrics there is a new (unique up to homotethy) Einstein metric.

2. The self-adjoint Ricci transformation.

If M is a connected manifold, a Riemannian metric on M is called homogeneous if the isometry group $I(M)$ acts transitively.

Let M be a manifold with a homogeneous metric of positive Ricci curvature. Then, the theorem of Bonnet-Myers implies that M is compact. Moreover, there exist a compact, connected and semisimple Lie subgroup of $I(M)$ acting transitively and effectively by isometries on M ([**DM**]). Thus,