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BANACH ALGEBRAS WITH UNITARY NORMS

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Dedicated to Edward G. Effros, for his pioneering contributions, on the occasion of his sixtieth birthday.

The metric nature of the unitary group of a (unital) C*-algebra is studied in a Banach-algebra framework.

1. Introduction and preliminaries.

The group $\mathfrak{A}_u$ of unitary elements in a (unital) C*-algebra $\mathfrak{A}$ is one of the critical structural components of $\mathfrak{A}$. From **[K]**, each U in $\mathfrak{A}_u$ is an extreme point of the unit ball $(\mathfrak{A})_1$ of $\mathfrak{A}$; in case $\mathfrak{A}$ is abelian, $\mathfrak{A}_u$ is precisely the set of extreme points of $\mathfrak{A}$. (More generally, when $\mathfrak{A}$ has a separating family of tracial states, $\mathfrak{A}_u$ is precisely the set of extreme points of $(\mathfrak{A})_1$.) Proceeding from this, Phelps **[P]** shows that the Krein–Milman property holds for $\mathfrak{A}_u$ and $(\mathfrak{A})_1$ when $\mathfrak{A}$ is abelian—namely, $(\mathfrak{A})_1$ is the norm closure of the convex hull $\mathrm{co}(\mathfrak{A}_u)$ of $\mathfrak{A}_u$. In **[RD]**, Dye and Russo remove the commutativity restriction—the Phelps result is valid for every unital C*-algebra. (This has become known as the "Russo–Dye theorem.") Gardner **[G]** gives a short and much simplified proof of the Russo–Dye theorem. A significant strengthening of the Russo–Dye theorem **[KP]** (based on a device in **[G]**) states that each A in $\mathfrak{A}$ with $\|A\| < 1$ is the (arithmetic) mean of a finite number of elements of $\mathfrak{A}_u$ —in finer detail, of n elements of $\mathfrak{A}_u$ when $\|A\| < 1 - \frac{2}{n}$ with $n = 3, 4, \ldots$. Haagerup **[H]** establishes a conjecture of Olsen and Pedersen **[OP]** by showing that this same is valid even when $\|A\| = 1 - \frac{2}{n}$ (a deep result).

Are these approximation properties of $\mathfrak{A}_u$ in $(\mathfrak{A})_1$ characteristic of C*-algebras? Is a Banach algebra $\mathfrak{A}$ with a subgroup $\mathfrak{G}$ of the group $\mathfrak{A}_{\mathrm{inv}}$ of invertible elements in $(\mathfrak{A})_1$ whose (norm-) closed, convex hull is $(\mathfrak{A})_1$ (isometrically, isomorphic to) a C*-algebra? We shall see that the answer to these questions is in the negative. In Section 4, we note that the Wiener algebra (functions with absolutely convergent Fourier series—equivalently, the group algebra $l_1(\mathbb{Z})$ of the additive group $\mathbb{Z}$ of integers) has the Russo–Dye (R–D) approximation property and is not isomorphic to a C*-algebra (even algebraically).