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# FOURIER MULTIPLIERS FOR $L_p(\mathbb{R}^n)$ VIA $q$–VARIATION

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**We give a new sufficient condition for a function to be a Fourier multiplier of $L_p(\mathbb{R}^n)$ via its $q$-variation on dyadic rectangles. This solves a problem posed by Coifman, Rubio de Francia and Semmes, who had considered the one-dimensional case.**

## 1. Introduction.

Let $I$ be an interval of $\mathbb{R}$. For $1 \le q < \infty$ we denote by $V_q(I)$ the space of all the complex-valued functions of bounded $q$-variation on $I$, that is, $V_q(I)$ consists of the functions $m$ on $I$ such that

$$\|m\|_{V_q(I)} = \sup \left( |m(x_0)|^q + \sum_{k \ge 0} |m(x_{k+1}) - m(x_k)|^q \right)^{1/q} < \infty,$$

where the supremum is taken over all increasing sequences $\{x_k\}_{k \ge 0}$ in $I$.

In [**2**], Coifman, Rubio de Francia and Semmes proved the following considerable improvement of the classical Marcinkiewicz multiplier theorem for $L_p(\mathbb{R})$.

**Theorem A.** *Let $I_k = [2^k, 2^{k+1}]$ and $J_k = [-2^{k+1}, -2^k]$ for every $k \in \mathbb{Z}$. Let $m \in L_\infty(\mathbb{R})$. If $\sup_{k \in \mathbb{Z}}(\|m\|_{V_q(I_k)} + \|m\|_{V_q(J_k)}) < \infty$ for some $1 \le q < \infty$, then $m$ is a Fourier multiplier for $L_p(\mathbb{R})$ for every $1 < p < \infty$ satisfying $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$.*

The ingredient of the proof of Theorem A in [**2**] is Rubio de Francia's generalized Littlewood-Paley inequality for arbitrary families of disjoint intervals (cf. [**7**]). Let us emphasize that the above theorem is one-dimensional, while the classical Marcinkiewicz theorem holds as well in the multiple dimensional case. The problem of extending Theorem A to $\mathbb{R}^n$ was left open in [**2**]. The purpose of this note is to solve it.

Let us define the space of functions of bounded $q$-variation on a rectangle of $\mathbb{R}^n$. We consider only rectangles with sides parallel to the axes, and also we restrict ourself to finite rectangles. Now let $R$ be such a rectangle. Write $R = \prod_{k=1}^n [a_k, b_k]$. Let $m$ be a function defined on $R$. Define $\Delta_R$ by

$$\Delta_R(m) = \Delta_{h_1}^{(1)} \Delta_{h_2}^{(2)} \cdots \Delta_{h_n}^{(n)} m(a_1, a_2, \cdots, a_n),$$