PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 2, 1996

ON CONSTRAINED EXTREMA

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Assume that I and J are smooth functionals defined on a Hilbert space H. We derive sufficient conditions for I to have a local minimum at y subject to the constraint that J is constantly $J(y)$.

The first order necessary condition for I to have a constrained minimum at y is that for some constant λ, $I'_y + \lambda J'_y$ is identically zero. Here I'_y and J'_y are the Fréchet derivatives of I and J at y. For the rest of the paper, we assume that y in H satisfies this necessary condition.

A common misapprehension (upon which much of the stability results for capillary surfaces has been based) is to assume that if the quadratic form $I''_y + \lambda J''_y$ is positive definite on the kernel of J'_y then I has a local constrained minimum at y. This is not correct in a Hilbert space of infinite dimension; Finn [**1**] has supplied a counterexample in the unconstrained case, and the same difficulty will occur in the constrained case. In the unconstrained case, if (as often occurs in practice) the spectrum of I''_y is discrete and 0 is not a cluster point of the spectrum, then I''_y positive definite at a critical point y implies that I''_y is strongly positive, (i.e., there exists $k > 0$ such that $I''_y(x) \geq k\|x\|^2$ holds for all x), and this in turn *does* imply that y is a local minimum (see [**2**]). However, in the constrained case, things are not so easy. Even if $I''_y + \lambda J''_y$ has a nice spectrum (in some sense), it is not clear that $I''_y + \lambda J''_y$ being positive definite on the kernel of J'_y implies that this quadratic form is strongly positive on the kernel, nor that strong positivity implies that y is a local minimum.

In [**3**], Maddocks obtained sufficient conditions for $I''_y + \lambda J''_y$ to be positive definite on the kernel of J'_y. As Maddocks points out, this is not quite enough to say that I has a constrained minimum at y. Remarkably, essentially the same conditions as Maddocks obtained for positive definiteness do in fact imply that I has a strict local minimum at y subject to the constraint $J = J(y)$, as we shall see.

For any $h \in H$ we may say $J(y+h) - J(y) = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$, where ϵ_1 goes to zero as $\|h\|$ goes to zero. If we consider an h for which $J(y+h) = J(y)$, then of course $0 = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$. Now, for