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SPECIALIZATIONS AND A LOCAL HOMEOMORPHISM THEOREM FOR REAL RIEMANN SURFACES OF RINGS

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Let $\phi : k \to A$ and $f : A \to R$ be ring morphisms, R a real ring. We prove that if $f : A \to R$ is étale, then the corresponding mapping between real Riemann surfaces $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$ is a local homeomorphism. Several preparatory results are proved, as well. The most relevant among these are: (1) a Chevalley's theorem for real Riemann surfaces on the preservation of constructibility via $\mathrm{S}_r(f)$, and (2) an analysis of the closure operator on real Riemann surfaces. Constructible sets are dealt with by means of a suitable first-order language.

1. Introduction.

Let k be a real ring. In this paper we study a sufficient condition for two real k-algebras A and R to have homeomorphic real Riemann surfaces. More precisely, here we show that if $f : A \to R$ is an étale morphism, then the corresponding mapping between real Riemann surfaces $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$ is a local homeomorphism (Theorem 9). In order to prove this theorem, we need several previous preparatory results, some of which are interesting on their own. Namely,

(a) the functorial character of S_r (Theorem 4),

(b) a Chevalley's theorem for real Riemann surfaces (Theorem 6), which guarantees that if f is finitely presented (in particular, if f is étale) then the image by $\mathrm{S}_r(f)$ of any constructible subset of $\mathrm{S}_r(R/k)$ is a constructible subset of $\mathrm{S}_r(A/k)$,

(c) a good knowledge of the closure operator on real Riemann surfaces (Theorem 1) and of the constructible subsets of real Riemann surfaces in terms of the first-order language of ordered valued fields,

(d) a result relating the notions of constructible, Tychonoff-closed, Tychonoff-clopen, closed and stability of a subset under specialization in real Riemann surfaces (Proposition 8) and, finally,

(e) the known result that if f is étale then $\mathrm{Spec}_r(f) : \mathrm{Spec}_r(R) \to \mathrm{Spec}_r(A)$ is a local homeomorphism. This theorem is due to M. Coste and M.-F. Roy.