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The weak dimension of algebras and its applications

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In this note we shall define the weak dimension of algebras Λ, analogous to the dimension of algebras in Cartan and Eilenberg [6], Ch. IX. In section 1 we shall characterize the algebras with the weak dimension zero, and study some properties of the weak dimension of the tensor product of two algebras, and we shall completely determine the weak dimension of fields. If an algebra Λ has a finite degree over a field K, it is well known that Λ is separable if and only if $\Lambda\otimes\Lambda^*$ $(=\Lambda^e)$ is semi-simple, where Λ^* is anti-isomorphic to Λ. Rosenberg and Zelinsky [15] proved that if Λ^e is a semi-simple algebra with minimum conditions, then $[\Lambda:K]<\infty$. Therefore if we want to define some generalized separability of algebras with infinite degree over K, then we may restrict ourselves to the case where Λ^e is semi-simple in the sense of Jacobson. In section 2 we shall call Λ R-separable if Λ^e is regular, and Λ has the property E, if $\Lambda\otimes L$ is regular for any field $L\geqq K$. We shall consider these algebras and relations between these two algebras. In section 3 we shall study some properties of tensor products of separable fields and algebras. In this note we always assume an algebra Λ has a unit element and that Λ-modules are unitary. We use [6] as a reference source for homological algebras.

1. The weak dimension of algebras

Let Λ be an algebra over a commutative ring K. We shall define the weak dimension of Λ (notation w. dim Λ), analogous to Cartan and Eilenberg [6], Ch. IX. 7.

DEFINITION 1. *w. dim Λ=the minimal integer n such that*

$$H_{n+1}(\Lambda, A)=Tor^{\Lambda^e}_{n+1}(A,\Lambda)=0$$

for any two sided Λ-module A.

First we state some remarks about the definition. Let Λ be an algebra over a field K. If Λ^e is Noetherian or if Λ is semi-primary with radical N such that $[\Lambda/N:K]<\infty$, then we have

$$\text{w. dim }\Lambda=\text{w. dim}_{\Lambda^e}\Lambda=\dim_{\Lambda^e}\Lambda=\dim\Lambda$$

from [6], Ch. VI, Exer. 3, and Auslander [2], Coro. 8 and [3], Th. 5.