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A generalization of a theorem of W. Hurewicz

By Jun-iti NAGATA

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W. Hurewictz proved the following theorem for separable metric spaces R and S.

If f is a closed continuous mapping of R onto S such that for each point q of S the inverse image $f^{-1}(q)$ consists of at most $m+1$ points, then $\dim S \leqq \dim R+m$[1].

This theorem was extended by K. Morita to $\operatorname{ind}\dim R$ and $\dim S$ of normal spaces R and S[2].

The purpose of this brief note is to generalize Hurewicz's theorem as follows.

THEOREM. *If f is a closed continuous mapping of a normal space R onto a perfectly normal space S such that for each point q of S the boundary $B(f^{-1}(q))$ of $f^{-1}(q)$ consists of at most $m+1$ points $(m \geqq 0)$, then*

$$\operatorname{ind}\dim S \leqq \operatorname{ind}\dim R+m\,.$$

Proof. We assume $\operatorname{ind}\dim R \leqq n$ and shall carry out the proof of $\operatorname{ind}\dim S \leqq n+m$ by induction with respect to $n \geqq -1$ and $m \geqq 0$.

1. This proposition is clearly valid for $n=-1$ and for every $m \geqq 0$.

2. Let us show the validity of this theorem for every $n>-1$ and for $m=0$. Assume G_1 and G_2 are arbitrary closed sets of S such that $G_1 \cap G_2=\phi$. Then $F_1=f^{-1}(G_1)$ and $F_2=f^{-1}(G_2)$ are disjoint closed sets of R. Hence we have, from $\operatorname{ind}\dim R \leqq n$, an open set U satisfying $F_1 \subseteq U \subseteq \bar{U} \subseteq F_2^c$[3], $\operatorname{ind}\dim(\bar{U}-U) \leqq n-1$. Since f is a closed mapping, $V=\{f(U^c)\}^c$ is an open set of S and it satisfies $G_1 \subseteq V \subseteq \bar{V} \subseteq G_2^c$. For $f^{-1}(G_1)=F_1 \subseteq U$ implies $G_1 \subseteq V$. $q \in G_2$ implies $f^{-1}(q) \subseteq F_2 \subseteq (\bar{U})^c$, and hence $(f(\bar{U}))^c=Q$ is an open nbd (=neighborhood) of q satisfying $Q \cap V=\phi$, proving $q \notin \bar{V}$ and consequently $\bar{V} \subseteq G_2^c$.

Letting $f(\bar{U}-U)=H$, we have a closed set H. Let q be an arbitrary point of $(\bar{V}-V)-H$; then $f^{-1}(q) \cap U \neq \phi$, $f^{-1}(q) \cap (\bar{U})^c \neq \phi$, $f^{-1}(q) \cap (\bar{U}-U)=\phi$. For $f^{-1}(q) \cap U =\phi$ implies $\{f(R-f^{-1}(q))\}^c=Q \ni q$, $Q \cap V=\phi$, i.e. $q \notin \bar{V}$. $f^{-1}(q) \cap (\bar{U})^c=\phi$ implies $q \in V$. The both cases are impossible. $f^{-1}(q) \cap (\bar{U}-U)=\phi$ is obvious.

We put $f^{-1}(q)_1=f^{-1}(q) \cap U$, $f^{-1}(q)_2=f^{-1}(q) \cap (\bar{U})^c$. Then we can show $B(f^{-1}(q)) \cap f^{-1}(q)_1 \neq \phi$. To show this we assume the contrary. Then $f^{-1}(q)_1$ is open,

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1) W. Hurewicz, Ein Theorem der Dimensionstheorie, Ann. Math., 31 (1930). We denote by $\dim R$ Lebesgue's dimension of R.

2) K. Morita, On closed mapping and dimension, Proc. Japan Acad., 32, no. 3 (1956). $\operatorname{Ind}\dim \phi=-1$, $\operatorname{ind}\dim R \leqq n$ if and only if for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq \bar{U} \subseteq G$, $\operatorname{ind}\dim(\bar{U}-U) \leqq n-1$.

3) We denote by F^c the complement set of F.