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On Kronecker products of primitive algebras

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In this note we shall prove some supplementary results of Jacobson [5] concerning Kronecker products of primitive algebras and those of P.M.I. algebras (that is, algebras with faithful minimal one sided ideals) and study their applications.

Let A_i ($i=1, 2$) be a primitive algebra over a field Φ and Δ_i be the division algebra of all A_i-endomorphisms of a faithful irreducible A_i-module (if A_i is a P.M.I. algebra, Δ_i is uniquely determined up to isomorphisms, and we shall call it the associated division algebra (denoted by *A.D.*) of A_i).

In section 1 we consider relations between semi-simplicity and primitivity of $A_1 \otimes A_2$ and those of $\Delta_1 \otimes \Delta_2$. In section 2, using results of section 1, for P.M.I. algebra A_i we study properties of Δ_i when $A_1 \otimes A_2$ is primitive or P.M.I., and give for a P.M.I. algebra A conditions under which $A \otimes A^*$ is primitive or P.M.I.. Further we prove that if B is central simple and $A \otimes B$ is P.M.I., then A is P.M.I. under special conditions. In section 3 we study the same problems as in section 2 in the case where primitivity is replaced by semi-simplicity. In section 4 we study Kronecker products of strongly dense algebras (see definition of section 4) and of closed irreducible algebras.

Throughout this note, we assume that algebras are all over a fixed ground field Φ, endomorphisms of right (left) A-module M act on the right (left) side of M, and that A^* means an anti-isomorphic algebra of an algebra A.

1. LEMMA 1. *Let A be a ring and e be an idempotent of A. If A is primitive (semi-simple in the sense of Jacobson [6]) then eAe is primitive (semi-simple). Further we assume that A is primitive, then A is a P.M.I. ring if and only if eAe is so.*

Proof. The first half is well known (cf. [5], Ch. 3, Pr. 7.1). Let A be a P.M.I. ring with the non zero socle $\mathfrak{S}$ and $\mathfrak{l}$ be a minimal left ideal of A such that $\mathfrak{l}e \neq 0$ for $\mathfrak{S}e \neq 0$, and $\mathfrak{l}e$ is a faithful minimal left ideal of A. For any non zero element exe of $e\mathfrak{l}e$, $eAe \cdot exe = e\mathfrak{l}e$ and $e\mathfrak{l}e$ is a faithful minimal left ideal of eAe, hence eAe is a P.M.I. ring. Conversely if A is primitive and eAe is a P.M.I. ring, then eAe has an idempotent e_0 such that e_0Ae_0 is a division ring, hence A is a P.M.I. ring.

PROPOSITION 1. *Let A be a right primitive algebra with a faithful irreducible module M and Δ be the associated division algebra of M, and let B be an algebra*