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Cohomology mod p of symmetric products of spheres

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Throughout this paper, we denote by $\mathfrak{S}_m$ the symmetric group of degree m, K a finite simplicial complex and p a fixed prime integer. The group $\mathfrak{S}_m$ operates in a natural way on the m-fold cartesian product $\mathfrak{X}_m(K)=K\times K\times\cdots\times K$. The orbit space $\mathfrak{S}_m(K)$ over $\mathfrak{X}_m(K)$ relative to $\mathfrak{S}_m$ is called the m-fold symmetric product. We study in the present paper the cohomology mod p of the symmetric product $\mathfrak{S}_m(S^n)$ of an n-sphere S^n. However the method we use will be applicable for calculation of cohomology of the symmetric product of more general complexes.

Let St^I denote the iterated Steenrod reduced powers, and $v_{0,m}$ a generator of $H^n(\mathfrak{S}_m(S^n);Z_p)\approx Z_p$. Then our main theorem is stated as follows[0]: If $q<n$ and $p^h\leqq m<p^{h+1}$, the vector space $H^{n+q}(\mathfrak{S}_m(S^n);Z_p)$ has a base formed by elements $\mathrm{St}^I v_{0,m}$, where I runs over the set of all admissible and special elements with degree q and length $\leqq h$. (See §3 for the precise definitions.)

The method we use is as follows.

Let $\mathfrak{S}_\infty(K)$ denote the infinite symmetric product of K. It follows from a result in my paper [7] that the injection homomorphism ι_m^*: $H^q(\mathfrak{S}_\infty(K);Z_p)\longrightarrow H^q(\mathfrak{S}_m(K);Z_p)$ is an epimorphism. As was proved by Dold-Thom [4], $\mathfrak{S}_\infty(K)$ is a product of the Eilenberg-MacLane complexes. Therefore we can describe a set of generators for $H^q(\mathfrak{S}_m(K);Z_p)$ in virtue of the Cartan's computation [2]. In order to examine if these generators are linearly independent, we choose a particular p-Sylow subgroup $\mathfrak{G}_m$ of $\mathfrak{S}_m$, and consider the orbit space $\mathfrak{G}_m(K)$ over $\mathfrak{X}_m(K)$ relative to $\mathfrak{G}_m$. The natural projection defines a homomorphism $\rho^*:H^q(\mathfrak{S}_m(K);Z_p)\to H^q(\mathfrak{G}_m(K);Z_p)$. We prove it by using of the transfer homomorphism that ρ^* is a monomorphism. Let $m=a_0p^h+a_1p^{h-1}+\cdots+a_h$ $(0\leqq a_i<p)$ be the p-adic expansion of m, and denote by $\mathfrak{Z}_p(K)$ the p-fold cyclic product of K (i.e. the orbit space over $\mathfrak{X}_p(K)$ relative to the subgroup $\mathfrak{Z}_p\subset\mathfrak{S}_p$ of cyclic permutations). Then we have that $\mathfrak{G}_m(K)$ is homeomorphic with the space $\mathfrak{X}_{a_0}(\mathfrak{Z}_p^h(K))\times\mathfrak{X}_{a_1}(\mathfrak{Z}_p^{h-1}(K))\times\cdots\times\mathfrak{X}_{a_h}(K)$, where $\mathfrak{Z}_p^r(K)$ denotes the iterated cyclic product $\mathfrak{Z}_p\mathfrak{Z}_p\cdots\mathfrak{Z}_p(K)$ (r-times) of K. As for the cohomology structure of $\mathfrak{Z}_p(K)$, I have studied in the paper [6]. By making use of some results there, we analyse the cohomology structure mod p of $\mathfrak{Z}_p^r(K)$, and we determine the dependence of the generators.

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0) (Added April 14, 1958) I have recently succeeded in determination of the cohomology ring $H^*(\mathfrak{G}_m(S^n);Z_p)$.