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CIRCLE PACKINGS ON COMPLEX AFFINE TORI

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1. Introduction

A set of closed disks on a plane is called a circle packing when they are arranged as follows. For each pair of distinct disks, they are disjoint or tangent. Moreover, the complement of the union of all disks is a disjoint union of triangular regions. Thus by taking the dual to the circle packing, we obtain a triangulation. We call its isotopy class a combinatorial type of the packing. A circle packing can be defined not only on a plane but also on a surface with a metric obviously by regarding a circle as a set of points with equal distance to the center. In this global case, a disk in the packing may be tangent to itself and a dual to the packing may be a triangulation in general sense. A surface will mean a surface with a reference homeomorphism (known as a marking) from a fixed surface throughout this paper.

It is not hard to show, and in fact will be shown in the third section that the set of flat tori, tori with Euclidean structures, which admit circle packings is dense in the space of all flat tori. However that set is known to be fairly poor by Andreev and Thurston. To describe this rigid property more precisely, recall that the space of flat tori up to similarity can be identified with the upper half plane $\mathbf{H}$. It was shown in [3] among other things that, to each triangulation, there is a unique flat torus up to similarity that admits a circle packing with a prescribed one as its combinatorial type. Then since the set of isotopy classes of triangulations on the torus is countable, only countably many flat tori admit circle packings in particular. This implies the fact, which we call Andreev-Thurston rigidity, that the flat structure on a torus with a packing does not admit any deformations which carry circle packings of the constant combinatorial type.

Let us consider a torus with another, more relaxed geometric structure in Thurston's sense on which the circle packing still makes sense. Namely we enlarge the transformation group to the complex affine group. The complex affine geometry is modeled on the complex plane $\mathbf{C}$ with the group of complex affine transformations, or equivalently similarity transformations. It is expressed in general as $z \mapsto az + b$ ($a, b \in \mathbf{C},\ a \neq 0$). Remark that by restricting the group to the Euclidean isometry, we obtain the flat geometry. Although there are no canonical metrics on an affine torus, transition maps are similar transformations and preserve the shapes of figures. Therefore we still can define circles on an affine torus. A circle on an affine torus is defined