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ALGEBRAIC REDUCTION OF TWISTOR SPACES OF HOPF SURFACES

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Introduction

Let Z be a compact connected complex manifold of dimension three. Let $a(Z)$ be the algebraic dimension of Z, i.e., the transcendence degree of the meromorphic function field of Z. We have $0 \leq a(Z) \leq 3$. When $a(Z) = 1$, Z admits a meromorphic map $f : Z \to Y$ onto a projective nonsingular curve Y, which induces by the pull-back an isomorphism of the meromorphic function fields. We call such a map f a *meromorphic algebraic reduction* of Z. Some restrictions on the possible structure of general fibers of f are known as in Ueno [10, 12.4, 12.5]. But the question as to which surfaces actually appear in such an algebraic reduction seems still open except in the Kähler case. (See [2] for the general results in the Kähler case.)

Now twistor spaces of compact anti-self-dual surfaces provide an interesting family of non-Kähler compact complex threefolds whose algebraic dimensions can take any values from zero to three. So it should be quite natural to consider the above question for these twistor spaces.

The purpose of this note is to answer the question in one of the simplest cases of the twistor spaces of Hopf surfaces, which, we hope, should serve as useful examples in understanding the situation in the general case. We show for instance that in case $a(Z) = 1$ either 1) a general fiber is a Hopf surface or 2) the normalization of a general fiber, which is in general non-normal, is a nonsingular ruled surface of genus one.

Note that Gauduchon [4] has already determined the algebraic dimensions of the twistor spaces of Hopf surfaces (cf. Proposition 1.1 below), generalizing the previous result of Pontecorvo [8], who has shown that the twistor spaces of special Hopf surfaces of the form $S(r, r)$ (cf. below) are of algebraic dimension two and has determined the structure of their algebraic reductions.

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1. Statement of Results

A *Hopf surface* is a compact analytic surface T whose universal covering is isomorphic to $W := \boldsymbol{C}^2 - \{0\}$. Suppose that T admits an anti-self-dual hermitian metric