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SEMICOLOCAL PAIRS AND FINITELY COGENERATED INJECTIVE MODULES

Dedicated to Professor Yukio Tsushima on his 60th birthday

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Let P and Q be rings, and ${}_PM$, N_Q and ${}_PV_Q$ a left P-module, a right Q-module and a P-Q-bimodule, respectively. Let $\varphi : M \times N \to V$ be a P-Q-bilinear map. Then we say that $({}_PM, N_Q)$ is a pair with respect to φ or simply a pair (see [12], [14], [10] or [1, Section 24]). For elements $x \in M$, $y \in N$ and for submodules ${}_PX \leq {}_PM$, $Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P-Q-subbimodule of ${}_PV_Q$ generated by $\{xy | x \in X, y \in Y\}$. A pair $({}_PM, N_Q)$ is said to be colocal if ${}_PMN_Q$ is colocal both as a left P-module and as a right Q-module. In [10] and [7], we studied colocal pairs related to some results in [5] and [4].

We shall define a semicolocal pair $({}_PM, N_Q)$ as a generalization of a colocal pair. A P-Q-bimodule ${}_PU_Q$ is said to be semicolocal if (i) the rings P and Q have complete sets $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ of orthogonal idempotents, respectively such that each e_iU_Q and each ${}_PUf_j$ are colocal modules and (ii) the socle of ${}_PU$ coincides with the socle of U_Q. Moreover a pair $({}_PM, N_Q)$ is said to be semicolocal if ${}_PMN_Q$ is semicolocal. Anh and Menini investigated semicolocal modules with some conditions related to duality (see [2]). In this note, we shall give some generalizations of results of [10] and [7] using the term "semicolocal pairs", and in particular give characterizations of finitely cogenerated injective modules (Theorems 2.4 and 2.5).

Throughout this note, P, Q and R are rings with identity and all modules are unitary. Let M be a module. Then $L \leq M$ ($L < M$) signifies that L is a (proper) submodule of M. By $\mathrm{S}(M)$, $\mathrm{T}(M)$ and $|M|$, we denote the socle, the top and the composition length of M, respectively. Moreover by $\mathrm{Pi}(R)$, we denote the set of primitive idempotents of R. Every homomorphism is written on the side opposite to the scalars.

1. Semicolocal pairs

A module M_R is said to be colocal if M_R has an essential simple socle.

Lemma 1.1. *Let f be an idempotent of R and M_R a colocal module with $\mathrm{S}(M_R) \cong \mathrm{T}(hR_R)$ for some $h \in \mathrm{Pi}(Q)$, where $Q = fRf$. Then Mf_Q is a colocal module with $\mathrm{S}(Mf_Q) = \mathrm{S}(M_R)f = \mathrm{S}(M_R)hQ$.*