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ON THE K-GROUPS OF SPHERICAL VARIETIES

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1. Statement of results

A spherical variety is a normal variety defined over a field with a split reductive group action with a dense open orbit isomorphic to a Borel subgroup. Flag varieties, Schubert varieties and toric varieties are examples of spherical varieties. In this paper we will study the K'-groups of varieties belonging to a certain category including spherical varieties. Our main results are descriptions of K'-groups and their coniveau filtrations of such varieties by means of their equivariant K'-groups. For a smooth toric variety, they are obtained by Morelli [4, Prop. 4]. Before we state our main results explicitly, we fix some notations.

Let B be a split connected solvable group defined over a field k. Then B is isomorphic to a product of an affine space and a torus as a variety over k. In this paper we are concerned with a B-variety X with finitely many B-orbits. All B-orbits of X are indexed by a finite set Δ. For $\sigma \in \Delta$, we denote by $\mathcal{O}(\sigma)$ the corresponding B-orbit of X. Let $M = \mathrm{Hom}(B, \mathbb{G}_m)$ be the character group of B. Any orbit $\mathcal{O}(\sigma)$ is isomorphic to a quotient scheme of B by a subgroup B_σ. Hence $\mathcal{O}(\sigma)$ is also isomorphic to a product of an affine space and a torus. Let $M^\sigma = \mathrm{Hom}(B_\sigma, \mathbb{G}_m)$, then M^σ becomes a quotient module of M.

Here we introduce K-theory. We denote by $K'_i(X)$ the i-th K-group of the category of coherent sheaves on X and by $K'_i(X, B)$ the i-th K-group of the category of B-equivariant coherent sheaves on X. Moreover we denote by $K_i(X)$ the i-th K-group of the category of locally free sheaves on X and by $K_i(X, B)$ the i-th K-group of the category of B-equivariant locally free sheaves on X.

In [6] R. Thomason showed that these two equivariant K-groups are isomorphic when X is smooth over k. The equivariant K-group of the base field $K_0(k, B)$ is isomorphic to the Grothendieck group of the category of k-representations of B. Hence we have $K_0(k, B) \simeq \mathbb{Z}[M]$. From this fact we can say that the equivariant K-group $K'_*(X, B)$ admits a $\mathbb{Z}[M]$-module structure. For a $\mathbb{Z}[M]$-module R, we denote by I_R the submodule of R generated by $\{rm - r; r \in R, m \in M\}$. The quotient module R/I_R is called the group of coinvariants of R and denoted by R_M.

We need an additional assumption on the characteristic of k. When B is not a torus, we assume $\mathrm{char} k = 0$. It is needed for varieties which we treat to admit a resolution of singularities.