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ERGODIC PROPERTIES OF RECURRENT SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

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Introduction

It is well known that for solutions of nonsingular finite-dimensional stochastic differential equations (and, more generally, for strongly Feller irreducible Markov processes in locally compact state spaces) there are only two possibilities of asymptotic behavior. If there exists a stationary distribution (that is, an invariant probability measure) then all solutions converge to it in distribution and the strong law of large numbers holds true. On the other hand, if there exists no stationary distribution then all solutions "escape to infinity in distribution" (in the sense specified by the formula (0.4) below). More specifically, consider a stochastic differential equation

$$\text{(0.1)} \qquad d\xi(t)=b(\xi(t))dt+\sigma(\xi(t))d\omega_t, \qquad \xi(0)=x\in \boldsymbol{R}^n,$$

in $\boldsymbol{R}^n$ where the coefficients b: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ and σ: $\boldsymbol{R}^n \to \boldsymbol{R}^{n\times n}$ are, for simplicity, globally Lipschitzian, ω_t is a standard n-dimensional Wiener process and $\sigma(y)\sigma^*(y)>0$, $y\in\boldsymbol{R}^n$. If there exists a stationary distribution μ for the equation (0.1) then

$$\text{(0.2)} \qquad \boldsymbol{P}_x\left[\frac{1}{T}\int_0^T \varphi(\xi(t))dt \to \int \varphi d\mu, T\to\infty\right]=1$$

holds for every x$\in\boldsymbol{R}^n$ and every μ-integrable function ϕ: $\boldsymbol{R}^n\to\boldsymbol{R}$, and

$$\text{(0.3)} \qquad P(t,x,A)\to\mu(A), \qquad t\to\infty$$

holds for $x\in\boldsymbol{R}^n$ and $A\in\mathscr{B}(\boldsymbol{R}^n)$ (the Borel sets on $\boldsymbol{R}^n$), where $P=P(t,x,A)$ stands for the transition probability function corresponding to the solutions of (0.1). If there exists no stationary distribution, then

$$\text{(0.4)} \qquad P(t,x,K)\to 0, \qquad t\to\infty$$