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ON EXISTENCE OF KÄHLER METRICS WITH CONSTANT SCALAR CURVATURE

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1. Introduction and Statements of Results

Let N be a compact Kählerian manifold, let Ω be a Kähler class on N, and let Ω^+ be the set of Kähler forms representing Ω. On Ω^+, consider the functional Φ_Ω that assigns to a Kähler form the square of the L^2-norm of the scalar curvature. A critical point of Φ_Ω is called an *extremal* Kähler metric. Any Kähler metric with constant scalar curvature is extremal. Conversely, the variational appraoch can be used to find metrics with constant scalar curvature.

We begin with an existence theorem for extremal metrics. Recall that a Kähler metric is called a *generalized Einstein-Kähler* metric if the eigenvalues of the Ricci tensor are constant, see [27]. For example, a product of Einstein-Kähler metrics is a generalized Einstein-Kähler metric. If M is homogeneous under the action of a compact Lie group, then every Kähler class on M is represented by a generalized Einstein-Kähler metric.

Theorem 1. *Let (M,g_M) be a generalized Einstein-Kähler manifold with non-negative Ricci curvature, and let (L,h) be a holomorphic Hermitian line bundle such that the eigenvalues of $c_1(L,h)$ with respect to g_M are constant on M. Suppose $\hat{L}$ is a Kählerian compactification of $L^0=(L\backslash zero\ section)$, and Ω is a Kähler class on $\hat{L}$ which is represented by a metric of 'special type'* (see Section 2). *Then $\hat{L}$ admits an extremal metric representing Ω. This metric is unique up to the action of the connected automorphism group* $\mathrm{Aut}^0(\hat{L})$.

This may be taken as a generalization of the existence theorem of Koiso and Sakane for Einstein-Kähler metrics, since if $c_1(\hat{L})>0$ and the Futaki character of $c_1(\hat{L})$ vanishes, then an extremal metric in the anticanonical class is necessarily Einstein. We interpret vanishing of a Futaki character as a condition that the scalar curvature of an extremal metric be constant, rather than as a condition for extendability of a