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ON NORMAL FORMS OF MODULAR CURVES OF GENUS 2

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0. Introduction

In this paper, we shall be interested in studying defining equations of algebraic curves X over $\bar{\boldsymbol{Q}}$, which are uniformized by arithmetic Fuchsian groups Γ.

It is well known that one can take the modular equation of level N, denoted by $\Phi_N(x, y)$, as a defining equation of the modular curve $X_0(N)$. This equation is very important, because it plays an essential role in complex multiplication theory over imaginary quadratic fields. Moreover it reflects a property of $X_0(N)$ as the coarse moduli space of generalized elliptic curves E with a cyclic subgroup of order N. However, in case of carrying out numerical calculations, it is difficult to treat the modular equation. The reason is that its degree and coefficients are fairly large. For example,

$$\begin{aligned}\Phi_2(x, y) = {} & x^3+y^3-x^2y^2+2^4\cdot 3\cdot 31xy(x+y)-2^4\cdot 3^4\cdot 5^3(x^2+y^2)+3^4\cdot 5^3\cdot 4027xy \\ & +2^8\cdot 3^7\cdot 5^6(x+y)-2^{12}\cdot 3^9\cdot 5^9\,, \\ \Phi_3(x, y) = {} & x^4+y^4-x^3y^3-2^2\cdot 3^3\cdot 9907xy(x^2+y^2)+2^3\cdot 3^2\cdot 31x^2y^2(x+y)+ \\ & 2^{15}\cdot 3^2\cdot 5^3(x^3+y^3)+2^{16}\cdot 3^5\cdot 5^3\cdot 17\cdot 263xy(x+y)+2\cdot 3^4\cdot 13\cdot 193\cdot \\ & 6367x^2y^2-2^{31}\cdot 5^6\cdot 22973xy+2^{20}\cdot 3^3\cdot 5^6(x^2+y^2)+2^{45}\cdot 3^3\cdot 5^9(x+y) \\ & \text{(cf. [8])}.\end{aligned}$$

Therefore it seems meaningful to give more convenient equations which can be treated easily and whose degrees and coefficients are as small as possible.

Suppose now that X is of genus two. Then the field $\bar{\boldsymbol{Q}}(X)$, consisting of rational functions on X defined over $\bar{\boldsymbol{Q}}$, is isomorphic to an algebraic function field $\bar{\boldsymbol{Q}}(x, y)$, where the relation between x and y is $y^2=f(x)$ and $f(T)\in\bar{\boldsymbol{Q}}[T]$ is a separable polynomial of degree 5 or 6. We call the equation $y^2=f(x)$ a normal form of X. In [2], Fricke determined normal forms of modular curves $X_0(23)$, $X_0(29)$, $X_0(31)$, which are sufficiently simple to treat easily from our viewpoint.

In this article, we will give the most efficient method for determining a normal