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ON *M*-RINGS AND GENERAL *ZPI*-RINGS

Dedicated to Professor Kentaro Murata on his 60th birthday

TAKASABURO UKEGAWA

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In the preceding paper [10], we have proved that a left Noetherian M-ring is a so called "general ZPI-ring" in the commutative case. Also we know that in an M-ring the multiplication of prime ideals is commutative [8]. In the present paper we define general ZPI-rings in section 1 and we study general properties of them, and as an important example of such rings we can give a left Noetherian semi-prime Asano left order. In section 2 we research the condition for a left Noetherian general ZPI-ring to be an M-ring, using minimal prime divisors of an ideal. The notation "$<$" means a proper inclusion as the preceding papers [8], [9], [10].

1. *M*-rings and general *ZPI*-rings

DEFINITION. If the multiplication of any two prime ideals of a ring R is commutative, and any ideal of R can be written as a produkt of powers of prime (considering R as a prime ideal) ideals of R, then we call R *a general ZPI-ring*. Therefore the multiplication of ideals is commutative.

In the commutative case a general ZPI-ring is necessarily Noetherian no matter whether the ring has an identity or not. But in our case the general ZPI-ring is not necessarily Noetherian as the example in [9] shows.

Proposition 1. *Let R be a left Noetherian general ZPI-ring, let P be any prime ideal of R, and let $\mathfrak{q}$ be maximal in the set of prime ideals such that $\mathfrak{q}<P$. Then for any ideal $\mathfrak{a}$ with $\mathfrak{q}<\mathfrak{a}<P$, there is an ideal $\mathfrak{b}$ such that $\mathfrak{a}=P\,\mathfrak{b}=\mathfrak{b}P$.*

Proof. Let $\mathfrak{a}=\mathfrak{p}_1\cdots\mathfrak{p}_r<P$, since R is a general ZPI-ring. Then $\mathfrak{p}_i\subseteq P$ for some $\mathfrak{p}_i$. Since $\mathfrak{q}<\mathfrak{a}\subseteq\mathfrak{p}_i$, $\mathfrak{q}<\mathfrak{p}_i\subseteq P$, so $\mathfrak{p}_i=P$. Therefore $\mathfrak{a}=P\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\mathfrak{p}_{i+1}\cdots\mathfrak{p}_r=\mathfrak{b}\,P$, where $\mathfrak{b}=\mathfrak{p}_1\cdots\mathfrak{p}_{i-1}\,\mathfrak{p}_{i+1}\cdots\mathfrak{p}_r$.

As in the commutative case we have

Proposition 2. *Let R be be a left Noetherian general ZPI-ring, and let P be a maximal ideal of R. Then there are no ideals between P and P^2 (including the case that $P=P^2$), more generally for any positive integer n, the only ideals*