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ON RIEMANNIAN MANIFOLDS ADMITTING CERTAIN STRICTLY CONVEX FUNCTIONS

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1. Introduction. Let M be an m-dimensional connected complete Riemannian manifold with metric g. For a smooth function f on M, the *Hessian* D^2f of f is defined by $D^2f(X,Y)=X(Yf)-D_XY\cdot f(X,Y\in TM)$. By a theorem of H.W. Wissner ([5; Satz. II. 1.3]), if there is a smooth function f on M such that $D^2f=g$ on M, then M is isometric to Euclidean space. In this note, we shall prove that if the Hessian of a smooth function f on M is close enough to g, then M is quasi-isometric to Euclidean space in the following sense: There exists a diffeomorphism $F: M\to R^m$ and some positive constant μ such that for each tangent vector X on M, $\mu^{-1}||X||_M\leqq||F_*X||_{R^m}\leqq||X||_M$. Our result contains the above theorem by Wissner as a special case ($\mu=1$), and generalises Yagi's theorem ([7]). Our theorem is stated as follows.

Theorem. *Let M be an m-dimensional connected complete Riemannian manifold with metric g. Suppose there exists a smooth function f on M which satisfies the following conditions*:

$$\text{(i)}\quad (1-H_1(f(x)))g(X,X)\leqq\frac{1}{2}D^2f(X,X)\leqq(1+H_2(f(x)))g(X,X)\,,$$

where $X\in T_xM(x\in M)$ and each H_i $(i=1,2)$ is a nonnegative continuous function on R,

$$\text{(ii)}\quad 1-H_1(t)>0 \textit{ for } t\in R \textit{ and } \lim_{t\to\infty}H_i(t)=0\ (i=1,2),$$

$$\text{(iii)}\quad \begin{cases}\displaystyle\int^\infty H_i(s)/s\,ds<+\infty\,,\\ \displaystyle\int^\infty\Big(\int_0^s H_i(u)du/s^2\Big)ds<+\infty\ (i=1,2)\,.\end{cases}$$

Then M is quasi-isometric to Euclidean space.

2. Proof of theorem and corollaries. Let M be an m-dimensional connected complete Riemannian manifold with metric g.

Lemma 1. *Let M and g be as above. Let f be a smooth function on M such*