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UNIRATIONAL QUASI-ELLIPTIC SURFACES IN CHARACTERISTIC 3

Dedicated to the memory of Taira Honda

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0. A non-singular projective surface X is called a *quasi-elliptic* surface if there exists a morphism $f: X\to C$, a curve, with almost all fibres irreducible singular rational curves E with $p_a(E)=1$ (cf. [4]). According to Tate [5], such surfaces can occur only in the case where the characteristic p of the ground field k is either 2 or 3, and almost all fibres E have single ordinary cusps. Let $\mathfrak{k}$ be the function field of C. Then the generic fibre of f with the unique singular point taken off is an elliptic $\mathfrak{k}$-form of the affine line $\boldsymbol{A}^1$ (cf. [2], [3]); if this form has a $\mathfrak{k}$-rational point(*) it is birational over $\mathfrak{k}$ to one of the following affine plane curves:

(i) If $p=3$, $t^2=x^3+\gamma$ with $\gamma\in\mathfrak{k}-\mathfrak{k}^3$.

(ii) If $p=2$, $t^2=x^3+\beta x+\gamma$ with $\beta,\gamma\in\mathfrak{k}$ and $\beta\notin\mathfrak{k}^2$ or $\gamma\notin\mathfrak{k}^2$.

On the other hand, if X is unirational C must be a rational curve. Conversely if C is a rational curve X is unirational. Indeed, $k(X)\otimes_{\mathfrak{k}}\mathfrak{k}^{1/3}$ is rational over k in the first case, and $k(X)\otimes_{\mathfrak{k}}\mathfrak{k}^{1/2}$ is rational over k in the second case. In this article we consider a unirational quasi-elliptic surface with a rational cross-section only in characteristic 3. Thus X is birational to a hypersurface $t^2=x^3+\phi(y)$ in the affine 3-space $\boldsymbol{A}^3$, where $\phi(y)\in\mathfrak{k}=k(y)$. If $\phi(y)$ is not a polynominal, write $\phi(y)=a(y)/b(y)$ with $a(y), b(y)\in k[y]$. Substituting t, x by $b(y)^3t$, $b(y)^2x$ respectively and replacing $\phi(y)$ with $b(y)^5a(y)$ we may assume that $\phi(y)\in k[y]$. Moreover, after making suitable birational transformations we may assume that $\phi(y)$ has no monomial terms whose degree are congruent to 0 modulo 3; especially that $d=\deg_y\phi$ is prime to 3. It is easy to see that under this assumption $f(x,y)=x^3+\phi(y)$ is irreducible.

A main result of this article is:

Theorem. *Let k be an algebraically closed field of characteristic* 3. *Then*

(*) This is equivalent to saying that f has a rational cross-section which is different from the section formed by the (movable) singular points of the fibres.