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EMBEDDING MANIFOLDS IN EUCLIDEAN SPACE

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1. Introduction. We consider here the problem of whether a smooth manifold M (compact, without boundary) embeds in Euclidean space of a given dimension. Our results are of two kinds: first we give sufficient conditions for an orientable n-manifold to embed in R^{2n-2}, and we then give necessary and sufficient conditions for RP^n ($=n-$dimensional real projective space) to embed in R^{2n-6}. We obtain these results using the embedding theory of A. Haefliger [6].

Recall that by Whitney [37], every n-manifold embeds in R^{2n}. Combining results of Haefliger [6], Haefliger-Hirsch [9] and Massey-Peterson [16] one knows that every *orientable* n-manifold embeds in R^{2n-1} ($n>4$), and if n is not a power of two, *every* n-manifold embeds in R^{2n-1}. Finally, if n is a power of two ($n>4$), by [9] and [26] one has: a non-orientable n-manifold embeds in R^{2n-1} if and only if $\bar{w}_{n-1}=0$. Here $\bar{w}_i, i\geqslant 0$, denotes the (mod 2) normal Stiefel-Whitney class of a manifold M.

We give two sets of sufficient conditions for embedding an n-manifold in R^{2n-2}; in order to use the theory of Haefliger, we assume $n\geqslant 7$.

Theorem 1.1. *Let M be an orientable n-manifold, with $\bar{w}_{n-3+i}=0$, for $i\geqslant 0$. If either $w_3\neq 0$, or $w_2\neq 0$ and $H_1(M; Z)$ has no 2-torsion, then M embeds in R^{2n-2}.*

Here w_i denotes the i^{th} mod 2 (tangent) Stiefel-Whitney class of M. A necessary condition for M^n to embed in R^{2n-2} is that $\bar{w}_{n-2}=0$. Note, however, that if $n-1$ is a power of two, then RP^n does not embed in R^{2n-2}, even though $\bar{w}_{n-2}=0$. (In this case $\bar{w}_{n-3}\neq 0$ and $H_1(RP^n; Z)=Z_2$).

By Massey-Peterson [16] one has that $\bar{w}_{n-3+i}=0$, $i\geqslant 0$, for M^n, provided one of the following conditions is satisfied: $n\equiv 3 \bmod 4$; $n\equiv 0, 2 \bmod 4$ and $\alpha(n)\geqslant 3$; $n\equiv 1 \bmod 4$ and $\alpha(n)\geqslant 4$. Here $\alpha(n)$ denotes the number of one's in the dyadic expansion of the integer n.

Recall that an orientable manifold is called a spin manifold if $w_2=0$. As a complement to Theorem (1.1) we have:

Theorem 1.2. *Let M be an n-dimensional spin manifold with $\bar{w}_{n-5+i}=0$,*

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