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ON THE PROJECTIVE CLASS GROUP OF FINITE GROUPS

Dedicated to Professor Kiiti Morita on his 60th birthday

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In this paper we will continue the investigation of integral representations of finite groups done in [3], [4] and [5]. We will here be concerned mainly with the projective class group of nilpotent and symmetric groups.

Let Σ be a (finite dimensional) semi-simple Q-algebra and let Λ be a Z-order in Σ. We will mean by the projective class group of Λ the class group defined by using all locally free, projective Λ-modules and denote it by $C(\Lambda)$.

Let Π be a finite group. A finitely generated Z-free Π-module is briefly called a Π-module. A Π-module is called a permutation Π-module if it can be expressed as a direct sum of $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence: $0\to M\to S\to S'\to 0$ where S and S' are permutation Π-modules.

As is well known, the projective class group $C(Z\Pi)$ of the group algebra $Z\Pi$ can be written as follows:

$$C(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi] \mid \mathfrak{A}(\neq 0) \text{ is a projective ideal of } Z\Pi\}\ .$$

We define the subgroups $\tilde{C}(Z\Pi)$, $C^q(Z\Pi)$ and $\tilde{C}^q(Z\Pi)$ of $C(Z\Pi)$ as follows:

$$\tilde{C}(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi) \mid \mathfrak{A}\oplus X\cong Z\Pi\oplus X \text{ for some } \Pi\text{-module } X\},$$

$$C^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi) \mid \mathfrak{A}\oplus S_1\cong S_2 \text{ for some permutation } \Pi\text{-modules } S_1 \text{ and } S_2\},$$

$$\tilde{C}^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi) \mid \mathfrak{A}\oplus S\cong Z\Pi\oplus S \text{ for some permutation } \Pi\text{-module } S\}\ .$$

Let Ω_Π be a maximal Z-order in $Q\Pi$ containing $Z\Pi$ and let $\psi_\Pi\colon C(Z\Pi)\to C(\Omega_\Pi)$ be the epimorphism induced by $\Omega_\Pi\underset{Z\Pi}{\otimes}\cdot$. Then the sequence $0\to\tilde{C}(Z\Pi)\to C(Z\Pi)\xrightarrow{\psi_\Pi}C(\Omega_\Pi)\to 0$ is exact.

In [3] and [4] we raised the following problem: